Piecewise Linear Differential Forms

Bachelor Thesis

Mathematics

by

Konrad Bals

Supervisor:

Prof. Dr. Thomas Nikolaus

Second Supervisor:

Prof. Dr. Johannes Ebert

Münster, 30.08.2019

0. Preface

The piecewise linear differential forms $\mathcal{A}_{\mathrm{PL}} X$ on a topological space X are a rational cochain complex with a multiplication map, such that its cohomology ring $H(\mathcal{A}_{\mathrm{PL}} X)$ coincides with the singular cohomology on X with coefficients in $\mathbb{Q}$. However, in contrast to the singular cochain complex C^*X of X, which can also be given a multiplicative structure, the multiplication on $\mathcal{A}_{\mathrm{PL}} X$ is always graded commutative.

In fact, from the existence of Steenrod squares on C^*X over finite prime fields one can deduce that there can be no natural commutative multiplication on the singular cochains with coefficients in $\mathbb{F}_p$ for a prime p or $\mathbb{Z}$ - or even no other commutative cochain algebra that is quasi-isomorphic. On the other hand, if the topological space X comes equipped with a structure of a smooth manifold, the de Rham complex Ω^*X of differential forms on X gives always a graded commutative cochain complex with real coefficients and by the de Rham theorem its cohomology and singular cohomology agree.

The piecewise linear differential forms now give a generalization of the construction of de Rham differential forms to arbitrary topological spaces; showing that there is a commutative model quasi-isomorphic to the singular cochain complex as soon as the coefficients are in a field with characteristic 0.

An application of the piecewise linear differential forms is an elegant approach to the homotopy theory of rational spaces due to Sullivan. But we will not be able to go into this theory here. The aim of this paper is rather to elaborate on the facts that are mentioned above and prove the most important properties of the piecewise linear differential forms.

In section 1 I give an introduction to the category of commutative differential algebra, the codomain category of $\mathcal{A}_{\mathrm{PL}}$. Section 2 is an introduction to the basic facts of simplicial sets and the definition of the polynomial differential forms $\mathfrak{A}^*_\bullet$, which will directly lead to the piecewise linear differential forms. To do this on an abstract level there is the section 3 purely about category theory with the purpose to give an explicit analysis of weak Kan extensions. Then section 4 can finally introduce the piecewise-linear differential forms and show that they not only give the same cohomology than singular cohomology, but also are quasi-isomorphic as differential graded algebras. We will see in section 5 that the same is true for the de Rham cohomology and proof that there is a quasi-isomorphism of differential graded algebras between $\mathcal{A}_{\mathrm{PL}} M$ and Ω^*M for smooth manifolds M. Finally, there is a first application given in section 6 by defining the notion of formality.

Furthermore, there are two topics dealt with in the appendix: Firstly in appendix A there is a quick introduction to de Rham differential forms on smooth manifolds with all statements about de Rham cohomology that we need in the later sections. Secondly, in appendix B there is a short encounter with Steenrod operations. They do not directly arise from piecewise linear differential forms, but they witness the non-commutativity of the singular cochain complex over $\mathbb{F}_p$ for a prime p - I will do the argument for $p = 2$. Hence they show that we cannot find a CDGA as $\mathcal{A}_{\mathrm{PL}} X$ for coefficients in finite prime fields.

It is important to mention that I do not make any claim of originality. All statements and proofs I give have already been known and written down by other people. This outline mostly relies on the book [FHT01] which gives a good and detailed introduction to most of these topics. The maps in the proofs of the two main theorems, i.e. the quasi-isomorphisms between $\mathcal{A}_{\mathrm{PL}}\,X$ and C^*X for a topological space X and between $\mathcal{A}_{\mathrm{PL}}\,M$ and Ω^*M for a smooth manifold M, and roughly the outline of the proofs basically com from this book.

They are given a bit more categorical flavor with ideas of Kan extensions and coends (and implicitly of model categories). The introduction to ends and coends and its relation to Kan extensions can be found in the very readable paper [Lor15]. Another good introduction to Kan extensions is [Leh14]. And for more general category theory in section 3 I have used the book [Rie14].

The implicit theorems about model categories appearing in section 2, i.e. $\mathcal{I}$-cell complexes and Kan fibrations come from the books [Hov07] and [GJ09].

Finally [Moe15] and [Moe17] both give many of the abstract arguments in the proofs concerning $\mathfrak{A}^*_\bullet$ and $\mathcal{A}_{\mathrm{PL}}$ in section 2 and 4 and in [Cam15] there are good geometric interpretations of piecewie-linear differential forms.

Another paper working with piecewise linear differential forms is [Hes06]. It gives our definition of CDGAs and has a section about formality of spaces.

For all general topology needs, especially together with [Mil89] for the appendix B about Steenrod squares, I have referred to [Bre93]. This book also has a section about de Rham cohomology which I have extended by the approach given in [MT97].

Contents

Throughout this paper let R be a commutative ring, from proposition 2.13 on we will even need R to be a field with characteristic 0. Furthermore, we fix the symmetric monoidal structure on the category $\mathbf{Ch}^*_R$ of cochain complexes over R given by the tensor product with Koszul sign convention, i.e. with braiding $B_{AB} : A^* \otimes_R B^* \to B^* \otimes_R A^*,\ a \otimes_R b \mapsto (-1)^{|a||b|} b \otimes_R a$ for all $A, B \in \mathbf{Ch}^*_R$ (we will often leave R implicit and just write $\otimes$).

1. Commutative Differential Graded Algebras

Definition 1.1. A *commutative differential graded algebra* (CDGA) is a commutative monoid in $\mathbf{Ch}_R^*$, i.e. a cochain complex (A^*, d) together with cochain maps

$$\mu : A^* \otimes_R A^* \to A^*;\ a \otimes b \mapsto a \cdot b, \qquad \eta : R \to A^*$$

such that μ is associative, graded commutative and satisfies

$$\mu(\eta \otimes Id_A) = Id_A = \mu(Id_A \otimes \eta).$$

The map μ is called multiplication und η unit of (A^*, d). We call (A^*, d) a *strict CDGA* if, moreover, μ is square zero in odd degree, i.e. $a^2 = 0$ for $a \in A^{\text{odd}}$.

A morphism of (strict) CDGAs $f : A^* \to B^*$ is a cochain map respecting μ and η, i.e. $f\mu_{A^*} = \mu_{B^*}(f \otimes f)$ and $f\eta_{A^*} = \eta_{B^*}$. Thus CDGAs over R form a category $\mathbf{CDGA}_R$ whith a full subcategory $\mathbf{CDGA}_R^{\text{strict}}$ given by strict CDGAs.

Remark 1.2. If 2 is a unit in R then obviosly every CDGA over R is strict. However over $\mathbb{F}_2$ the polynomial algebra $\mathbb{F}_2[x]$ is a CDGA which is not strict.

The differential on a CDGA A^* satisfies the Leibnitz rule. Indeed, as μ is a cochain map the definition of the differential $d_\otimes$ on $A^* \otimes A^*$ yields

$$d(a \cdot b) = \mu(d_\otimes(a \otimes b)) = da \cdot b + (-1)^{|a|} a \cdot db$$

Remark 1.3. For a CDGA A^* we also get a CDGA structure on $A^* \otimes A^*$ making the multiplication μ on A^* a CDGA map. Here we really need that A^* is commutative.

It follows from the definition of the category $\mathbf{CDGA}_R^{(\text{strict})}$ that there is a forgetful functor from $\mathbf{CDGA}_R^{(\text{strict})}$ to $\mathbf{Ch}_R^*$, suggesting that there should be a kind of free (strict) CDGA over a cochain V^*. This is indeed true and we will be interested in the strict part:

Proposition 1.4. *Given an R-module V we have a commutativ graded algebra (CGA) defined as*

$$\Lambda V := TV/J$$

where TV denotes the tensor algebra of V and $J \trianglelefteq TV$ is the ideal generated by $a \otimes b - (-1)^{|a||b|} b \otimes a$ for all $a, b \in TV$ and a^2 for all $a \in TV^{\text{odd}}$. Moreover this CGA is free, in the sense that it satisfies the following universal property for all strict CGAs A^*:

$$\begin{array}{ccc} V & \longrightarrow & A^* \\ \downarrow & \nearrow_{\exists!} & \\ \Lambda V & & \end{array}$$

□

By this universal property of ΛV every map of strict CGAs is already defined by the restriction to V, as is every derivation. This implies that given a cochain structure (V, d) on V we get a unique strict CDGA structure on ΛV. We denote this strict CDGA by $(\Lambda V, d)$ and have:

Proposition 1.5. *The strict CDGA* $(\Lambda V, d)$ *is free in* $\mathbf{CDGA}_R^{strict}$ *over the cochain* (V, d). □

Sometimes it is convenient to have a different description of ΛV: Differing between even and odd degrees let us establish:

$$J^{even} = \langle a \otimes b - b \otimes a, |a| \cdot |b| \text{ even} \rangle, \qquad J^{odd} = \langle a \otimes b + b \otimes a, |a| \cdot |b| \text{ odd} \rangle$$

and hence by definition of the symmetric algebra $S(V)$ and exterior algebra $E(V)$ of V we get:

$$\Lambda V = (T^{even}(V)/J^{even}) \otimes (T^{odd}(V)/J^{odd}) = S^{even}(V) \otimes E^{odd}(V)$$

2. Simplicial Objects

We want to define simplicial objects for arbitrary categories, in a way how we have defined the set of singular simplices for topological spaces.

Definition 2.1. The category $\mathbf{\Delta}$ of *finite ordinal numbers* consists of finite, non-empty, linear-ordered sets as objects and order preserving maps as morphisms.

For every object $\Delta \in \mathbf{\Delta}$ (note that the object Δ is not written in bold!) there is a unique isomorphism $\Delta \to \Delta_n := \{0, \ldots, n\}$ for a unique $n \in \mathbb{N}$. Then it turns out that any order preserving map between finite ordinals is a composition of the maps

$$d_i^n : \Delta_n \to \Delta_{n+1}; d_i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

omitting i and called the *ith face map* and

$$s_i^n : \Delta_n \to \Delta_{n-1}; s_i(j) = \begin{cases} j & j < i \\ i & j = i \\ j-1 & j > i \end{cases}$$

hitting i twice and called *ith degeneracy* map. Furthermore the maps d_i^n, s_i^n satisfy the *simplicial identities*:

$$d_j^{n+1} d_i^n = d_i^{n+1} d_{j-1}^n, \quad \text{for } i < j, \qquad s_j^n s_i^{n+1} = s_i^n s_{j+1}^{n+1}, \quad \text{for } i \leq j$$

$$s_j^{n+1} d_i^n = \begin{cases} d_{i-1}^{n-1} s_j^n & j < i-1 \\ Id_{\Delta_n} & j = i-1, j = i \\ d_i^{n-1} s_{j-1}^n & j > i \end{cases}$$

Definition 2.2. A *simplicial object* in a category $\mathcal{C}$ is a functor from $\mathbf{\Delta}^{op}$ to $\mathcal{C}$. We will denote a simplicial object by $X_\bullet$ and $X_n := X_\bullet(\Delta_n)$. These objects together with natural transformations as morphisms form a category called $s\mathcal{C}$.

According to the last observation, defining a simplicial object $X_\bullet$ is equivalent to specifying for each Δ_n an object X_n with maps $d_i^n : X_n \to X_{n-1}$ and $s_i^n : X_n \to X_{n+1}$ satisfying the simplicial identities.

Example 2.3. For $\mathcal{C} = \mathbf{Set}$ we have already defined some simplicial sets in the context of simplices on a topological space. For a space X the set of singular simplices $\mathrm{Sing}_\bullet(X)$ is a simplicial set via

$$\mathbf{\Delta}^{op} \xrightarrow{|-|} \mathbf{Top}^{op} \xrightarrow{\mathbf{Top}(-,X)} \mathbf{Set}$$

where $|-| : \mathbf{\Delta} \to \mathbf{Top}$ denotes the geometric realization functor $\Delta \mapsto |\Delta| = S_1^1(0) \cap \mathbb{R}_{\geq 0}^{\oplus\Delta} \subset \mathbb{R}^{\oplus\Delta}$ with the subspace topology and $S_1^1(0)$ is the 1-norm sphere centered in 0 with radius 1. One can check that the induced functions under the free functor $\Delta \mapsto \mathbb{R}^{\oplus\Delta}$ restrict correctly to the subset $S_1^1(0) \cap \mathbb{R}_{\geq 0}^{\oplus\Delta}$ and geometric realization is indeed functorial.

Shortly we will come back to simplicial sets and look at them in more detail because they will play a key role in this paper. But as mentioned before we are also interested in a simplicial CDGA:

2.1. Polynomial Differential Forms

Definition 2.4. The *algebra of polynomial forms* $\mathfrak{A}_\bullet^*$ is the simplicial object in **CDGA** defined by the assignment

$$\Delta \mapsto \mathfrak{A}_\Delta^* := \Lambda \left(\langle\Delta\rangle_R^\vee \xrightarrow{\text{id}} \langle\Delta\rangle_R^\vee \right) \Big/ J_\Delta$$

where $\langle\Delta\rangle_R^\vee$ is the dual space to the free R-module over Δ sitting in degree 0 and 1 and J_Δ is the ideal generated by $1 - \sum \Delta$ in degree 0 and $\sum \Delta$ in degree 1 (here we identify Δ with the corresponding basis of $\langle\Delta\rangle_R^\vee$).

To be more concrete we obtain the following description of the objects $\mathfrak{A}_n^* = \mathfrak{A}_{\Delta_n}^*$.

$$\mathfrak{A}_n^* := (\Lambda(t_0, \dots, t_n, dt_0, \dots, dt_n)/J_n, d)$$

where $|t_i| = 0$ with obvious differential and $J_n \trianglelefteq V$ generated by $\{1 - \sum_{i=0}^n t_i, \sum_{i=0}^n dt_i\}$. Furthermore one can explicitly compute the face and degeneracy maps

$$d_i^n : \mathfrak{A}_n^* \to \mathfrak{A}_{n-1}^* : t_k \mapsto \begin{cases} t_k & k < i \\ 0 & k = i \\ t_{k-1} & k > i \end{cases}$$

$$s_i^n : \mathfrak{A}_n^* \to \mathfrak{A}_{n+1}^* : t_k \mapsto \begin{cases} t_k & k < i \\ t_k + t_{k+1} & k = i \\ t_{k+1} & t > i \end{cases}$$

Remark 2.5. At this point it is probably good to quickly look at the geometry of these objects and shed light on the name polynomial differential forms.

Recall (or confer appendix A) that in the setting of smooth manifolds a differential form is a smooth section in the exterior algebra of the cotangent bundle of a manifold M. This is a CDGA $\Omega^*(M)$ with the $\wedge$-product as multiplication and every differential form has the form:

$$\omega = \sum f_k dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

for $f_k \in C^\infty(M) = \Omega^0(M)$ and $dx_i \in \Omega^1(M)$ the standard projections to the i-th coordinate. Now this is almost exactly how the elements in $\mathfrak{A}_n^*$ look like:

$$\mathfrak{a} = \sum p_k(t_0, \ldots, t_n) \cdot dt_{i_1} \cdots dt_{i_k}$$

for $p_k(t_0, \ldots, t_n) \in \mathfrak{A}_n^0 = R[t_0, \ldots, t_n]/(\sum t_i - 1)$ a polynomial and dt_i generators in degree 1. In the cases that $R \subset \mathbb{R}$ is a subring, $R[t_0, \ldots, t_n]$ is a subset of $C^\infty(\mathbb{R}^{n+1})$. The condition $\sum t_i = 1$ now restricts the polynomials to functions defined on the geometric realization of the standard n-Simplex Δ_n and we get $\mathfrak{A}_n^0 \subset C^\infty(|\Delta_n|)$. Moreover, the tangent space(s) of $|\Delta_n|$ are given by the $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ "parallel" to $|\Delta_n|$, i.e. such that $dx_0 + \cdots + dx_n = 0$, hence the second half of the quotient in $\mathfrak{A}_n^*$. It follows now that modulo renaming we have an inclusion $\mathfrak{A}_n^* \subset \Omega^*(|\Delta_n|)$ of CDGAs and as graded algebras the equality

$$\Omega^*(|\Delta_n|) = C^\infty(|\Delta_n|) \otimes_{\mathfrak{A}_n^0} \mathfrak{A}_n^*.$$

This justifies the name polynomial differential forms.

2.2. Simplicial sets

To better understand polynomial forms and later piecewise linear forms we have to look a bit more at simplicial sets. By definition a simplicial set $K \in \mathbf{sSet}$ can be identified as a collection of sets K_n with face and degeneracy maps. Via the Yoneda embedding $Y : \mathbf{\Delta} \to \mathbf{sSet}$ we get standard simplicial sets $\Delta[n] := Y(\Delta_n)$ and as a direct consequence of the Yoneda lemma we have $K_n = \mathbf{sSet}(\Delta[n], K)$.

There are two other important simplicial sets: The boundary $\partial\Delta[n]$ is defined as

$$\partial\Delta[n] := \bigcup d_i \Delta[n-1] \subset \Delta[n]$$

having all non-degenerate simplices of $\Delta[n]$ exept for $\mathrm{id} : \Delta_n \to \Delta_n$. And we have the kth horn $\Lambda^k[n]$ of $\Delta[n]$ as

$$\Lambda^k[n] := \bigcup_{i \neq k} d_i \Delta[n-1] \subset \partial\Delta[n]$$

having the non-degenerate simplices of $\partial\Delta[n]$ except for $d_k : \Delta_{n-1} \to \Delta_n$ omitting k. Geometrically one can think of $\partial\Delta[n]$ as $\Delta[n]$ without the interior and $\Lambda^k\Delta_n$ as $\Delta[n]$ without the interior and the face opposite to the vertex r.

Now we can also give a geometrical-like description of a simplicial set in the following sense:

Definition 2.6. Let $\mathcal{C}$ be a category with all small colimits and $\mathcal{I} \subset \mathrm{Mor}(\mathcal{C})$ a set of morphisms in $\mathcal{C}$. Then *$\mathcal{I}$-cell* is the smallest set of morphisms containing $\mathcal{I}$ that is closed under (transfinite) composition, coproducts, pushouts along arbitrary maps and sequential colimits. An *$\mathcal{I}$-cell object* is an object $c \in \mathcal{C}$ such that the unique morphism from the initial object $\emptyset \to c$ is in $\mathcal{I}$-cell.

Example 2.7. Relative CW-complexes are by definition relative $\mathcal{I}$-cell complexes in **Top** with $\mathcal{I} = \{S^{n-1} \hookrightarrow D^n : n \geq 0\}$.

Theorem 2.8. *In the category* **sSet** *with* $\mathcal{I} = \{\partial\Delta[n] \to \Delta[n]\}$ *a map is in $\mathcal{I}$-cell if and only if it is injective.*

Proof. Firstly, to be injective is a point-wise property of a map in **sSet**, also the colimit constructions in **sSet** are point-wise. Now it is straight forward to see that in **Set** injectivity is closed under composition, pushouts and sequential colimits. Because $\partial\Delta[n] \to \Delta[n]$ is injective the last observation yields that every map in $\mathcal{I}$-cell is injective.

For the other direction let $K \hookrightarrow L$ be an injective map. We will inductively define simplicial sets X^n such that $K \to X^n$ is in $\mathcal{I}$-cell and approximates $K \hookrightarrow L$. Set $X^{-1} = K$ and let X^{n-1} already be constructed such that $X^{n-1} \hookrightarrow L$ is a bijection on X^{n-1}_m for $m \leq n-1$. Now we denote the set of all n-simplices in $L_n \setminus X^{n-1}_n$ with S_n; because $X^{n-1} \hookrightarrow L$ is a map between simplicial sets and by construction of X^{n-1} these n-simplices have to be non-degenerate. By the remark above we can identify every object in S_n with a simplicial map $\Delta[n] \to L$ and its restriction to $\partial\Delta[n]$ factors uniquely through X^{n-1}. We now consider the following pushout diagram:

$$\begin{array}{ccc} \coprod_{S_n} \partial\Delta[n] & \longrightarrow & X^{n-1} \\ \Big\downarrow & \ulcorner & \Big\downarrow \\ \coprod_{S_n} \Delta[n] & \longrightarrow & X^n \end{array}$$

The induced map $X^n \to L$ is onto all m-simplices for $m \leq n$ because $X^{n-1} \hookrightarrow L$ is for $m \leq n-1$ and every n-simplex of L is hit by construction. Furthermore $X^n \to L$ is still one-to-one because again $X^{n-1} \hookrightarrow L$ is and all simplices in S_n are non-degenerate. Now by construction $L = \underrightarrow{\operatorname{colim}}\, X^n$ and the map $K \to \underrightarrow{\operatorname{colim}}\, X^n = L$ lies in $\mathcal{I}$-cell and agrees with $K \hookrightarrow L$ as needed. □

Corollary 2.9. *Every simplicial set is an $\mathcal{I}$-cell complex with* $\mathcal{I} = \{\partial\Delta[n] \to \Delta[n]\}$. □

If K is a simplicial set we call the X^n from theorem 2.8 skeleta of K and write $\mathrm{sk}_n(K) := X^n$.

There is another interesting set of morphisms in **sSet** which are in some sense dual to $\mathcal{I}$-cell. (This duality is made precise in the theory of model structures on categories.)

Definition 2.10. A *Kan fibration* is a map in **sSet** that has the right lifting property against all horn inclusions. That means $K \to L$ is a Kan fibration if given a solid diagram

$$\begin{array}{ccc} \Lambda^k\Delta_n & \longrightarrow & K \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & L \end{array}$$

there exists the dashed arrow. A map is called a *trivial Kan fibration* if it has the right lifting property against the inclusion $\partial\Delta[n] \to \Delta[n]$. Finally a simplicial set K is called a *(trivial) Kan complex* if the unique arrow $K \to *$ is a (trivial) Kan fibration.

Returning to the polynomial differential forms we can now prove a very important property. Forgetting the extra structure of a simplicial CDGA the underlying simplicial set of $\mathfrak{A}^p_\bullet$ is a trivial Kan complex for every p. In fact every simplicial group has automatically the horn filling property and is therefore a Kan complex. But instead of proving this general statement, which is not hard to do but quite technical, we will do the triviality part and Kan complex part at once.

Lemma 2.11. *We have an extra "degeneracy-like" map $s : \mathfrak{A}^p_{n-1} \to \mathfrak{A}^p_n$ for every p given by $s(1) = (1-t_n)^2$ and $s(t_i) = (1-t_n)t_i$ satisfying $d_n s = \mathrm{id}$ and $d_i s = s d_i$ for $i < n$.*

Proof. This is just a straightforward calculation:

$$d_n s(1) = d_n(1-t_n)^2 = (1-0)^2 = 1 \qquad d_n s(t_i) = d_n(1-t_n)t_i = (1-0)t_i = t_i$$

And because $s : \mathfrak{A}^p_{n-2} \to \mathfrak{A}^p_{n-1}$ is given by the formula abouve with t_n replaced by t_{n-1} we get for $i < n$:

$$\begin{aligned} d_i s(1) &= d_i(1-t_n)^2 = (1-t_{n-1})^2 = s d_i(1) \\ d_i s(t_j) &= d_i(1-t_n)t_j = (1-t_{n-1})d_i(t_j) = s d_i(t_j). \end{aligned}$$

□

If we already knew that $\mathfrak{A}^p_\bullet$ were a Kan complex, we could directly use this extra "degeneracy-like" map s to show that $\mathfrak{A}^p_\bullet$ is a trivial Kan complex. But we are going to do both at the same time in the next proposition.

Proposition 2.12. *The sCDGA $\mathfrak{A}^p_\bullet$ as a simplicial set is a trivial Kan complex for every p. That is in every solid diagram as below, there exists the dashed lift:*

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \mathfrak{A}^p_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & * \end{array}$$

Proof. Let $x_0, \ldots, x_n \in \mathfrak{A}^p_{n-1}$ the images of the non-degenerated $(n-1)$-simplices of $\partial\Delta[n]$ via the map $\partial\Delta[n] \to \mathfrak{A}^p_\bullet$. In particular $d_i x_j = d_{j-1} x_i$ for $i < j$. We will inductively define elements $u^k \in \mathfrak{A}^*_n$, such that for all $0 \le i \le k$ we have $d_i u^k = x_i$.

Set $u^0 = s_0 x_0$. Now for $0 < k < n-1$ define a new object $v^k = s_{k+1}\big((d_{k+1}u^k)^{-1}x_{k+1}\big)$. We have for $i \le k$:

$$d_i v^k = d_i s_{k+1}\Big((d_{k+1}u^k)^{-1}x_{k+1}\Big) = s_k\Big((d_k d_i u^k)^{-1} d_i x_{k+1}\Big) = s_k\big((d_k x_i)^{-1}\underbrace{d_i x_{k+1}}_{=d_k x_i}\big) = 1$$

and $d_{k+1}v^k = (d_{k+1}u^k)^{-1}x_{k+1}$. Letting $u^{k+1} = u^k v^k$ we get as far as u^{n-1}. Note that getting from u^{n-1} to u^n to finish the proof we would need an "extra degeneracy map s_n", which doesn't exist. However plugging the map $s : \mathfrak{A}^p_{n-1} \to \mathfrak{A}^p_n$ from lemma 2.11 into the formula for v^{n-1} the calculations above still hold and we obtain our element u^n as the desired lift. □

This is basically the general proof for simplicial groups and one can see that without an "extra degeneracy-like" map there is only the chance of filling horns. The technical part then is to work around the one omitted face.

Now we get to a very important fact about rational polynomial forms. Therefore in the next part of this paper we need stronger assumptions on R: From now R will always be a field of characteristic 0.

Proposition 2.13. *The sCDGA $\mathfrak{A}^*_n$ over R is acyclic as a cochain complex for every $n \ge 0$, i.e. $H(\mathfrak{A}^*_n) = R$.*

Proof. For the cohomology calculations we consider defining $\mathfrak{A}^*_n$ via n variables and no relation instead of $(n+1)$-variables and one relation. That is

$$\mathfrak{A}^*_n = \Lambda(t_0, \ldots, t_n, y_0, \ldots, y_n)/J_n \cong \Lambda(t_1, \ldots, t_n, y_1, \ldots, y_n) = \Lambda(t_1, y_1) \otimes \cdots \otimes \Lambda(t_n, y_n)$$

We can easily compute the cohomology of $\Lambda(t_i, y_i)$. The only nontrivial degrees are 0 and 1 and we have $d(t_i^k) = k t_i^{k-1} y_i$ as the differential.

$$\left\langle t_i^0, t_i^1, t_i^2, \ldots \right\rangle \xrightarrow{d} \langle t_i^0 y_i, , t_i^1 y_i, t_i^2 y_i \ldots \rangle \to 0 \to \cdots$$

Over fields d is surjective, such that the first cohomology vanishes and in characteristic 0 the kernel of d is precisely generated by $t_i^0 = 1$, i.e. given by R. Thus $H(\Lambda(t_i, y_i)) = R$ is the graded ring concentrated in degree 0. Finally the Künneth formula over fields yields

$$H(\mathfrak{A}^*_\bullet) = H(\Lambda(t_1, y_1)) \otimes \cdots \otimes H(\Lambda(t_n, y_n)) = R \otimes \cdots \otimes R = R \qquad \square$$

This proposition will play a key role in our later proof and it is false if we are not over a field of characteristic 0.

3. A bit category theory

So far we have only worked with simplicial objects, i.e. with functors from $\mathbf{\Delta}^{op}$ to another category. Our goal is to extend these functors to a bigger domain. For example we would like to somehow associate polynomial forms to topological spaces and not only to elements $\Delta \in \mathbf{\Delta}$. There is an abstract machinery from category theory called Kan extensions that exactly does this. For this we first have to introduce a little end and coend calculus.

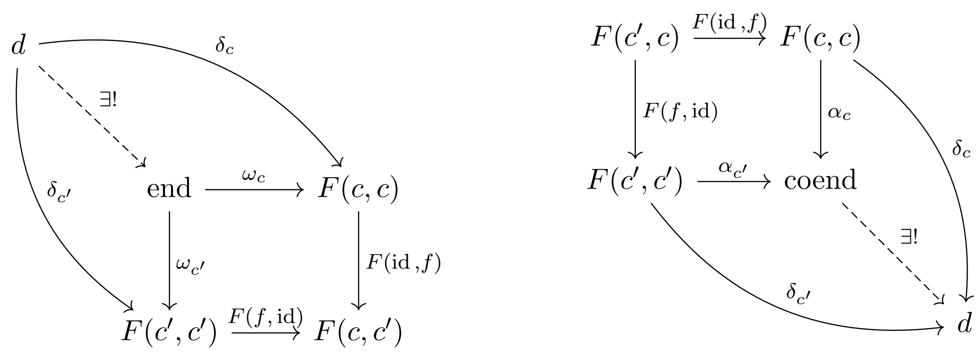
3.1. Ends and Coends

Let $F : \mathcal{C}^{op} \times \mathcal{C} \to D$ a bifunctor. A *wedge* of F is an object $d \in D$ with maps $\delta_c : d \to F(c,c)$, such that for all $f : c \to c'$ the following diagram commutes:

$$\begin{array}{ccc} d & \xrightarrow{\delta_c} & F(c,c) \\ \big\downarrow{\scriptstyle \delta_{c'}} & & \big\downarrow{\scriptstyle F(\mathrm{id},f)} \\ F(c',c') & \xrightarrow{F(f,\mathrm{id})} & F(c,c') \end{array}$$

Analogously a *cowedge* is a wedge in $\mathcal{D}^{op}$.

Definition 3.1. Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ a functor. An *end* of F is final wedge (end, ω) of F and a *coend* of F is an initial cowedge (coend, α) of F. In pictures this means



for every co/wedge (d, δ) and $f : c \to c'$.

By abuse of notation forgetting all maps we often refer to the object in $\mathcal{D}$ as an end or coend and denote it by $\int_{\mathcal{C}} F(c,c)$ and $\int^{\mathcal{C}} F(c,c)$ respectively.

We have a very nice consequence of this explicit description of ends and coends:

Proposition 3.2. *The following diagrams are equalizer respectively coequalizer diagrams:*

$$\int_{\mathcal{C}} F(c,c) \xrightarrow{\ \omega\ } \prod_{\mathcal{C}} F(c,c) \underset{F_*}{\overset{F^*}{\rightrightarrows}} \prod_{f:c\to c'} F(c,c')$$

$$\coprod_{f:c\to c'} F(c',c) \underset{F_*}{\overset{F^*}{\rightrightarrows}} \coprod_{\mathcal{C}} F(c,c) \xrightarrow{\ \alpha\ } \int^{\mathcal{C}} F(c,c)$$

with F_* and F^* having the covariatn resp. contravariant induced morphisms from $f : c \to c'$ as in definition 3.1 on the component indexed by $f : c \to c'$. Especially continuous functors preserve ends and cocontinuous functors preserve coends.

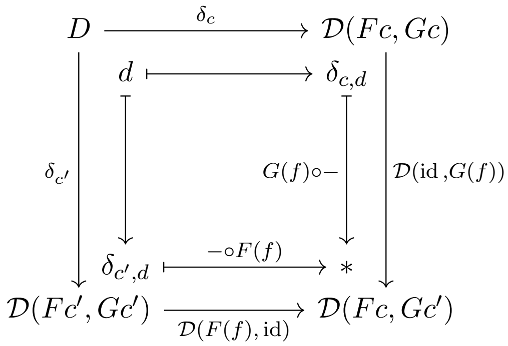
We will need the following strong link between ends and natural transformation.

Proposition 3.3. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, then we have an isomorphism*

$$\mathrm{Nat}(F, G) \cong \int_{\mathcal{C}} \mathcal{D}(Fc, Gc) \tag{3.1}$$

with the wedge maps $\omega_c : \mathrm{Nat}(F, G) \to \mathcal{D}(Fc, Gc); \eta \mapsto \eta_c$.

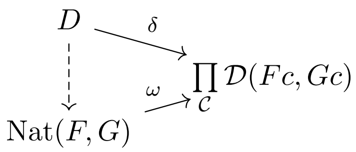
Proof. Let (D, δ) be a wedge of the functor $\mathcal{D}(F-, G-) : \mathcal{C} \to \mathbf{Set}$. Then D is a set and for any $d \in D$ we have a map $\delta_{c,d} := \delta_c(d) : Fc \to Gc$ such that



commutes for some fixed $* : Fc \to Gc'$. The inner square reformulated states that $\delta_{c,d}$ is natural in c and we obtain a map

$$D \to \mathrm{Nat}(F, G), \quad d \mapsto \delta_{-,d}$$

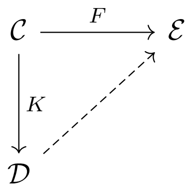
which is unique in respect to the diagram



Therefore $\mathrm{Nat}(F, G)$ satisfies the universal property and by uniqueness of ends we get the isomorphism $\mathrm{Nat}(F, G) \cong \int_{\mathcal{C}} \mathcal{D}(Fc, Gc)$. □

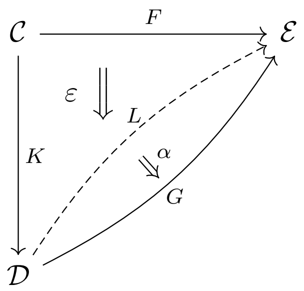
3.2. Kan Extensions

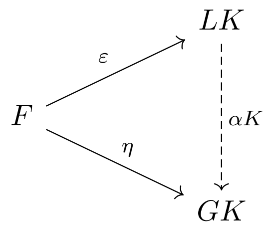
Now we can start to introduce the Kan extensions to extend our functors that we have defined so far. But beside our application of Kan extensions they also play a central role in general category theory generalizing limits, colimits, adjoints. Given two functors



we want to find a functor $\mathcal{D} \to \mathcal{E}$ which is closest to commuting in the diagram. That motivates the following definition:

Definition 3.4. In the situation above a tuple (L, ε) consisting of a functor $L : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\varepsilon : F \Rightarrow L \circ K$ is called a *pointwise left Kan extension*, if for all functors $G : \mathcal{D} \to \mathcal{E}$ with natural transformations $\eta : F \Rightarrow G \circ K$ there is a unique transformation $\alpha : L \Rightarrow G$ with $\eta = \varepsilon K$. In diagrams it says, starting with the left diagram





there exists a unique natural transformation $\alpha : R \Rightarrow G$, such that the right diagram commutes. Dually one defines the *pointwise right Kan extension* with all natural transformations reversed.

It is important to note that neither left nor right Kan extensions need to exist, but if they do they are unique up to unique isomorphism. Therefore we will write $\operatorname{Lan}_K F$ for the left and $\operatorname{Ran}_K F$ for the right Kan extension of F along K.

Proposition 3.5. *Let $K : \mathcal{C} \to \mathcal{D}$ be a functor, such that all left Kan extensions along functors $\mathcal{C} \to \mathcal{D}$ exist. Then the functor $\operatorname{Lan}_K : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$ is left adjoint to the restriction functor $- \circ K$.*

Proof. This is just a reformulation of the universal property that for any natural transformation $F \Rightarrow G \circ K$ there is a unique transformation $\operatorname{Lan}_K F \Rightarrow G$:

$$\operatorname{Nat}(F, G \circ K) \cong \operatorname{Nat}(\operatorname{Lan}_K F, G)$$ □

This motivates a slightly different definition:

Definition 3.6. A *weak left Kan extension* along $K : \mathcal{C} \to \mathcal{D}$ is a functor $[\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{D}]$ which is left adjoint to the restriction functor $- \circ K$. Dually the *weak right Kan extension* is the right adjoint of the restriction functor. If there is no confusion we also write Lan_k and Ran_k for the left resp. right Kan extension.

The last proposition shows that if all pointwise Kan extensions along a fixed functor exist, then they are the weak Kan extensions. But a weak Kan extension need not be a pointwise Kan extension as the commutativity of the right diagram above is a real demand. However, from now on we will only consider weak Kan extensions and remark that in all our settings they are indeed pointwise.

In order to get our hands on Kan extensions via coends we have to introduce another new notion:

Definition 3.7. Let $c \in \mathcal{C}$ be an object and $S \in \mathbf{Set}$ a set. The *tensor* or *copower* of F with S denoted $S \cdot F$ is the S-indexed coproduct of c, i.e. $S \cdot c = \coprod_S c$. This defines a bifunctor $\mathbf{Set} \times \mathcal{C} \to \mathcal{C}$ satisfying

$$\mathcal{C}(S \cdot c, d) \cong \mathbf{Set}(S, \mathcal{C}(c, d)) \tag{3.2}$$

Dually we can define the *cotensor* or *power* of F with S denoted by F^S as the copower in $\mathcal{C}^{op}$.

The reason for **Set** appearing in this definition is that we live in a **Set**-enriched world. In general for a $\mathcal{V}$-enriched category one can define a tensor by substituting **Set** by $\mathcal{V}$ in the last sentence.

Remark 3.8. From the definition of the tensor it follows directly that $(- \cdot c)$ is left adjoint to the hom-functor $\mathcal{C}(c, -)$. In particular it commutes with all colimits.

Theorem 3.9. *Given functors $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ we have*

$$\mathrm{Lan}_K F \cong \int^{\mathcal{C}} \mathcal{D}(Kc, -) \cdot Fc$$

Proof. This turns out to be a straight forward calculation:

$$\begin{aligned} \mathrm{Nat}\left(\int^{\mathcal{C}} \mathcal{D}(Kc, -) \cdot Fc, G\right) &\overset{(3.1)}{\cong} \int_D \mathcal{E}\left(\int^{\mathcal{C}} \mathcal{D}(Kc, d) \cdot Fc, Gd\right) \\ \cong \int_D \int_{\mathcal{C}} E(\mathcal{D}(Kc, d) \cdot Fc, Gd) &\overset{(3.2)}{\cong} \int_{\mathcal{C}} \int_D \mathbf{Set}(\mathcal{D}(Kc, d), \mathcal{E}(Fc, Gd)) \\ \overset{(3.1)}{\cong} \int_{\mathcal{C}} \underbrace{\mathrm{Nat}(\mathcal{D}(Kc, -), \mathcal{E}(Fc, G-))}_{\cong \mathcal{E}(Fc, GKc) \text{ by Yoneda}} &\cong \int_{\mathcal{C}} \mathcal{E}(Fc, GKc) \overset{(3.1)}{\cong} \mathrm{Nat}(F, GK) \end{aligned}$$

proving the adjointness. □

Corollary 3.10. *If $\mathcal{E}$ is a cocomplete category, then all left Kan extensions of functors with codomain $\mathcal{E}$ exist. Analogously the same is true for right Kan extensions if $\mathcal{E}$ is complete.* □

We can even say more about Kan extensions if the functor $K : \mathcal{C} \to \mathcal{D}$ is fully faithful:

Proposition 3.11. *For $K : \mathcal{C} \to \mathcal{D}$ fully faithful the natural transformation $F \Rightarrow \operatorname{Lan}_K F \circ K$ is an isomorphism*

Proof. With our new formula at hand we can just check:

$$\operatorname{Lan}_K F(Kx) = \int^{\mathcal{C}} \mathcal{D}(Kc, Kx) \cdot Fc \cong \int^{\mathcal{C}} \mathcal{C}(c, x) \cdot Fc = \operatorname{Lan}_{\mathrm{id}} F(x) = F(x) \qquad \square$$

Let $\mathcal{P}(\mathcal{C}) := [\mathcal{C}^{op}, \mathbf{Set}]$ be the presheaf category of $\mathcal{C}$ and denote by $Y : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ the Yoneda embedding. By the Yoneda lemma Y is a fully faithful functor and from the last proposition we get the co-Yoneda lemma:

Proposition 3.12 (co-Yoneda lemma)**.** *The representable presheaves of $\mathcal{C}$ are dense in the presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$, i.e. for any presheaf d we have $d = \varinjlim d_i$ for $d_i \in Y(\mathcal{C})$.*

Proof. We have the isomorphism

$$Fx \cong \operatorname{Lan}_{\mathrm{id}} Fx = \int^{\mathcal{C}} \mathcal{C}(c, x) \cdot Fc.$$

In the special case that the tensor product is between two sets A, B we have

$$A \cdot B = \coprod_A B \cong A \times B = B \times A \cong \coprod_B A = B \cdot A$$

and we can rewrite $Fx = \int^{\mathcal{C}} Fc \cdot \mathcal{C}(c, x)$. Keeping in mind that a coend is actually a colimit we see, that F is a colimit of representable functors $\mathcal{C}(c, -)$, i.e. of elements $d_i \in Y(\mathcal{C})$. □

This proposition morally says that the presheaf category of $\mathcal{C}$ is a colimit-completion of $\mathcal{C}$. We will even be able to show that the presheaf category is universal with this property. But before we can do that we will need another lemma about the Yoneda embedding:

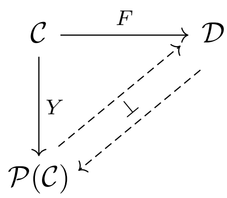
Lemma 3.13. *The Left Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ has a right adjoint, in particular it preserves colimits.*

Proof. Putting all the information about ends, coends, tensors and the Yoneda lemma together we can compute with $X \in \mathcal{P}(\mathcal{C})$ and $d \in \mathcal{D}$.

$$\begin{aligned}\mathcal{D}\left(\int^{\mathcal{C}} \mathcal{P}(\mathcal{C})(Yc, X) \cdot Fc, d\right) &\cong \int_{\mathcal{C}} \mathcal{D}(Xc \cdot Fc, d) \overset{(3.2)}{\cong} \int_{\mathcal{C}} \mathbf{Set}(Xc, \mathcal{D}(Fc, d)) \\ &\overset{(3.1)}{\cong} \mathcal{P}(\mathcal{C})(X, \mathcal{D}(F, d)).\end{aligned}$$

Thus the functor $\mathcal{D} \to \mathcal{P}(\mathcal{C})$ given by $d \mapsto \mathcal{D}(F-, d)$ is right adoint to the left Kan extension. □

In pictures we have



Example 3.14. The functor Sing : **Top** → **sSet** from example 2.3 was defined via the geometric realization functor as $X \mapsto \mathbf{Top}(|-|, X)$. Now we can see that it is actually the right adjoint of the left Kan extension of the geometric realization functor $|-|$ along the Yoneda embedding. Furthermore this left Kan extension is called *geometric realization*:

$$\mathbf{sSet} \to \mathbf{Top}; \qquad K \mapsto \int^n K_n \cdot |\Delta_n|$$

Theorem 3.15 (Yoneda Advanced)**.** *Let $F : \mathcal{C} \to D$ a functor and $\mathcal{D}$ cocomplete. Then we have an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{-\circ Y} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

from the category of cocontinuous functors $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ to the category of functors $\mathcal{C} \to \mathcal{D}$ given by restriction along the Yoneda embedding.

Proof. We will show that Lan_Y is the inverse functor. Thus, we have to find natural isomorphisms $1 \Rightarrow \mathrm{Lan}_Y(-) \circ Y$ and $1 \Rightarrow \mathrm{Lan}_Y(- \circ Y)$. Because of the Yoneda lemma Y is fully faithful and proposition 3.11 gives us isomorphisms $F \Rightarrow \mathrm{Lan}_Y F \circ Y$ natural in F. Hence the first natural transformation.

On the other hand with $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ cocontinous we have for $d \in Y(C)$

$$\mathrm{Lan}_Y(F \circ Y)(d) = \int^{\mathcal{C}} \mathcal{P}(\mathcal{C})(Yc, d) \cdot F \circ Yc \cong \int^{x \in Y(C)} \mathcal{P}(\mathcal{C})(x, d) \cdot Fx = F(d).$$

The general case follows from the co-Yoneda lemma and the last lemma. Because every element in $\mathcal{P}(\mathcal{C})$ is of the form $\varinjlim d_i$ for $d_i \in Y(C)$ we get:

$$\mathrm{Lan}_Y(F \circ Y)(\varinjlim d_i) \cong \varinjlim \mathrm{Lan}_Y(F \circ Y)(d_i) \cong \varinjlim F(d_i) \cong F(\varinjlim d_i).$$

Therefore, also the other direction holds and we have the equivalence of categories. □

It is the independence of the choice of a colimit representation in the last equation that needed most of the hard work so far.

Remark 3.16. We finally have the universal property of the presheaf category of $\mathcal{C}$ as the colimit-completion of $\mathcal{C}$, i.e. all functors $F : \mathcal{C} \to \mathcal{D}$ into a cocomplete category factor uniquely as cocontinuous functors through the Yoneda embedding.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle Y}\downarrow & \nearrow_{\exists!\,\mathrm{colim}} & \\ \mathcal{P}(\mathcal{C}) & & \end{array}$$

4. Piecewise linear differential forms

We can finally return to our object of interest and apply our new techniques. Because **CDGA** is cocomplete all left Kan extensions exist and we can apply our theorems from the last section.

Definition 4.1. We define the functor $\mathcal{A}_{\mathrm{PL}} : \mathbf{sSet} \to \mathbf{CDGA}^{op}$ of piecewise linear differential forms via

$$\mathcal{A}_{\mathrm{PL}} = \mathrm{Lan}_Y(\mathfrak{A}^*_\bullet)$$

For a topological spaces we extend this to get a functor

$$\mathcal{A}_{\mathrm{PL}} : \mathbf{Top} \to \mathbf{CDGA}^{op}; \quad X \mapsto \mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}_\bullet X) := \mathcal{A}_{\mathrm{PL}}(X)$$

Remark 4.2. Following lemma 3.13 we will directly get a right adjoint $K_\bullet : \mathbf{CDGA}^{op} \to \mathbf{sSet}$ via

$$K_n = \mathbf{CDGA}^{op}(\mathfrak{A}^*_n, -) = \mathbf{CDGA}(-, \mathfrak{A}^*_n).$$

One can show that the adjunction $\mathcal{A}_{\mathrm{PL}} \dashv K_\bullet$ is a Quillen adjunction for the standard model structures on **sSet** and **CDGA**. Therefore the functor $\mathcal{A}_{\mathrm{PL}}$ transfers the homotopy theory form simplicial sets (which themselves are Quillen equivalent to topological spaces) to **CDGA**.

Remark 4.3. As in the cases of the polynomial differential forms, it is not entirely clear, why firstly this should be a notion of differential forms and secondly why they are piecewise linear. If we have a look at the formula of the left Kan extension we get in **CDGA**:

$$\mathcal{A}_{\mathrm{PL}}(K) = \left(\int^n \mathbf{sSet}(\Delta[n], K) \cdot \mathfrak{A}^*_n \right)^{op} = \int_n {\mathfrak{A}^*_n}^{\mathbf{sSet}(\Delta[n],K)} \cong \int_n {\mathfrak{A}^*_n}^{K_n}$$

because the dual of the tensor is the power and the dual of the coend is an end. As a set forgetting structure we have ${\mathfrak{A}^*_n}^{K_n} = \mathbf{Set}\left(*, {\mathfrak{A}^*_n}^{K_n}\right)$. However, by defining addition, multiplication and the differential object-wise we can retrieve our structure from $\mathbf{Set}\left(*, {\mathfrak{A}^*_n}^{K_n}\right)$. Now with the defining property of the power we can continue calculating:

$$\int_n {\mathfrak{A}^*_n}^{K_n} \cong \int_n \mathbf{Set}\left(*, {\mathfrak{A}^*_n}^{K_n}\right) \overset{(3.2)^{op}}{\cong} \int_n \mathbf{Set}(K_n, \mathbf{Set}(*, \mathfrak{A}^*_n)) \overset{(3.1)}{\cong} \mathbf{sSet}(K, \mathfrak{A}^*_\bullet)$$

Therefore we have $\mathcal{A}_{\mathrm{PL}}(K) = \mathbf{sSet}(K, \mathfrak{A}^*_\bullet)$ as a set with object-wise CDGA-structure. Meaning a piecewise linear differential form is a choice of a polynomial differential form for every simplex, such that this choice commutes with the face and degeneracy maps. (As a matter of fact, we could have defined $\mathcal{A}_{\mathrm{PL}}(K)$ this way, but we wouldn't have the theory behind this definition.)

Now as a comparison to smooth manifolds a differential form was a smoothly varying assignment of a multilinear map to every point, such that it becomes a smooth function on the entire manifold, i. e. is compatible with chart transitions. Thus if one thinks of the geometric realization of a simplicial set as a simplicial complex, these differential forms $\mathcal{A}_{\mathrm{PL}}$ are piecewise polynomial on a piecewise linear topological space. Piecewise polynomial would have been a more appropriate name, but piecewise-linear is somehow common in the literature.

Proposition 4.4. *The functor* $\mathcal{A}_{\mathrm{PL}} : \mathbf{sSet} \to \mathbf{CDGA}$ *sends injective morphisms to surjective morphisms.*

Proof. By theorem 2.8 every injective simplicial map is obtained by transfinite composition, pushouts and sequential colimits starting from $\partial\Delta[n] \to \Delta[n]$. Because $\mathcal{A}_{\mathrm{PL}}$ as a left Kan extension is contravariant cocontinuous it sends pushouts to pullbacks and sequential colimits to inverse limits. Thus we only have to check, that surjectivity is stable under these three operations und verify the statement for $\mathcal{A}_{\mathrm{PL}}(\Delta[n]) \to \mathcal{A}_{\mathrm{PL}}(\partial\Delta[n])$.

The last claim translates to

$$\mathbf{sSet}(\Delta[n], \mathfrak{A}^p_\bullet) \to \mathbf{sSet}(\partial\Delta[n], \mathfrak{A}^p_\bullet)$$

is surjective for every p. This is just the lifting property against $\partial\Delta[n] \to \Delta[n]$ which is satisfied because $\mathfrak{A}^p_\bullet$ is a trivial Kan complex.

For the remaining stability statements we first have a look at **Set**, where it is clear that pullbacks and inverse sequential limits of surjective maps are surjective. Then because the forgetful functor $\mathbf{CDGA} \to \mathbf{Set}$ is right adjoint it preserves limits and the underlying set of a limit in **CDGA** is the limit of the underlying sets. Finally surjectivity in **CDGA** is just a property of the underlying sets and, indeed, it was enough to check the claim on sets. □

4.1. The equivalence $\mathcal{A}_{\mathrm{PL}}$ and C^*

Our next goal is to find a zigzag of quasi-isomorphisms

$$\mathcal{A}_{\mathrm{PL}}\, X \to \bullet \leftarrow C^* X$$

for all spaces X between the piecewise linear differential forms on X and the singular cochain complex of X. We would like to think of both functors as going from **sSet** to $\mathbf{DGA}^{op}$. So we intruce the simplicial cochain functor $C^* : \mathbf{sSet} \to \mathbf{DGA}^{op}$ that satisfies $C^* X = C^*(\mathrm{Sing}\, X)$.

Definition 4.5. The *simplicial chain functor* is the composite

$$C_* : \mathbf{sSet} \xrightarrow{\langle - \rangle_R} \mathbf{sMod}_R \xrightarrow{M_*} \mathbf{Ch}_*$$

of the free functor from simplicial sets to simplicial R-modules and the Moore complex functor M_* that assign to every simplicial module $A_\bullet$ the chain complex $M_n(A_\bullet) = A_n$ with differential

$$\partial = \sum_{i=0}^{p+1} (-1)^{i+1} d_i$$

With this definition we can also define the *simplicial cochain functor* $C^* : \mathbf{sSet} \to \mathbf{DGA}^{op}$ by taking the dual of the Moore complex and defining a multiplication, i.e. for a simplicial set K we have

$$C^n(K) = \mathbf{Mod}_R(\langle K_n \rangle_R, R) \cong \mathbf{Set}(K_n, R)$$

with differential for $f \in C^p(K)$ (note the Koszul sign konvention) and multiplication for $f, g \in C^*(K)$ given by

$$df = \sum_{i=0}^{p+1} (-1)^{p+i+1} f d_i, \qquad f \cup g = (f \otimes g)\Delta_{AW}$$

where Δ_{AW} is the Alexander-Whitney diagonal approximation.

Moreover we can even further restrict our domain category: We have by construction of C^* and the fact that the free functor $\langle - \rangle_R$ commutes with colimits and $\mathbf{Mod}_R(-, R)$ sends colimits to limits

$$C^*(\underrightarrow{\operatorname{colim}}\, K_i) \cong \underleftarrow{\lim}\, C^* K_i.$$

The right hand side is now a colimit in $\mathbf{DGA}^{op}$ and hence C^* preserves colimits. Therefore theorem 3.15 yields that C^* is the left Kan extension of its restriction along the Yoneda embedding Y. We denote this simplicial DGA as

$$C^*_\bullet : \ \mathbf{\Delta} \xrightarrow{\ Y\ } \mathbf{sSet} \xrightarrow{\ C^*\ } \mathbf{DGA}^{op}$$

and have $C^*_n = C^*(\Delta[n])$. We will show that the canonical inclusions make

$$\mathfrak{A}^*_\bullet \longrightarrow \mathfrak{A}^*_\bullet \otimes C^*_\bullet \longleftarrow C^*_\bullet$$

a chain of quasi-isomorphisms of trivial Kan complexes and that this is sufficient to deduce that the left Kan extensions are quasi-isomrophic as well. Thus we want to prove:

Lemma 4.6. *1) The simplicial DGA $C^p_\bullet$ as a simplicial set is a trivial Kan complex for every p and $H(C^*_n) = R$ for all $n \geq 0$.*

2) The simplicial DGA $\mathfrak{A}^p_\bullet \otimes C^q_\bullet$ *is as a simplicial set a trivial Kan complex for all* p, q *and* $H(\mathfrak{A}^*_\bullet \otimes C^*_\bullet) = R$ *for all* $n \geq 0$ *again.*

Proof. Ad 1): Given a map $\partial\Delta[n] \to C^p_\bullet$ let $f_i \in C^p_{n-1} = \mathbf{Set}(\Delta[n-1]_p, R)$ the images of the non-degenerated $(n-1)$-simplices, i.e.

$$\partial_i f_j = \partial_{j-1} f_i \qquad i < j. \tag{4.1}$$

We want to find a cochain $f : \Delta[n]_p \to R$ having the f_i's as boundaries. First identify each f_i with a map from $d_i\Delta[n-1]_p \subset \Delta[n]_p$ to R via $f_i' : d_i\Delta[n-1]_p \to R$ such that $f_i' d_i = f_i$ and our condition (4.1) translates to

$$f_j' d_j d_i = f_j d_i = d_i f_j \overset{(4.1)}{=} d_{j-1} f_i = f_i d_{j-1} = f_i' d_i d_{j-1} = f_i' d_j d_i$$

for all $i < j$. That just means that f_i' and f_j' coincide for every element in $d_j d_i(\Delta[n-2]_p) = d_i(\Delta[n-1]_p) \cap d_j(\Delta[n-1]_p)$ and we can glue these maps together to obtain a map $f' : \bigcup d_i\Delta[n-1]_p \to R$. Let f be the extension of f' to entire $\Delta[n]_p$ vanishing outside $\bigcup d_i\Delta[n-1]_p$. Then $f \in C^p_n$ and for all $i \in \mathcal{I}$ we have $d_i f = f d_i = f_i$.

We will show the second statement for simplicial homology. Applying UCT then gives the proof in cohomology. Let $[s_{i_0}, \dots, s_{i_k}]$ denote the map in $\Delta[n]_k$ given by $k \mapsto s_k$. Now we define the chain contraction

$$\tau : C_k(\Delta[n]) \to C_{k+1}(\Delta[n]), [s_{i_0}, \dots, s_{i_k}] \mapsto \begin{cases} [s_0, s_{i_0}, \dots, s_{i_k}] & s_0 \in [s_{i_0}, \dots, s_{i_k}](\Delta_k) \\ 0 & \text{else} \end{cases}$$

A short calculation for $\sigma = [s_{i_0}, \dots, s_{i_k}] \in C_k(\Delta[n])$ and $k > 0$ yields

$$d\tau\sigma = \begin{cases} \sum_{i=0}^{k+1} (-1)^i d_i [s_0, s_{i_0}, \dots, s_{i_k}] = \sigma - \tau d\sigma & s_0 \in [s_{i_0}, \dots, s_{i_k}](\Delta_k) \\ 0 & \text{else} \end{cases}$$

and $d\tau\sigma[s_{i_0}] = [s_{i_0}] - [s_0]$. Therefore τ is a chain homotopy between $\mathrm{id} : C_*(\Delta[n]) \to C_*(\Delta[n])$ and the composition $C_*(\Delta[n]) \xrightarrow{\epsilon} R \to C_*(\Delta[n])$ given by $\sigma \mapsto [s_0]$ in degree 0 and 0 otherwise. That means $C_*(\Delta[n])$ is chain homotopic to R and the claim follows.

Ad 2): Because $\mathfrak{A}^p_\bullet$ and $C^q_\bullet$ are already trivial Kan complexes we can also find a lift of $\partial\Delta[n] \to \Delta[n]$ into $\mathfrak{A}^p_\bullet \otimes C^q_\bullet$: We consider the $\mathbf{Set}$-isomorphism

$$f : \mathfrak{A}^p_{n-1} \otimes C^q_{n-1} \to \mathbf{Set}(\Delta[n-1]_q, \mathfrak{A}^p_{n-1}); \qquad \sum_\alpha \Phi_\alpha \otimes f_\alpha \mapsto \sum_\alpha f_\alpha(-)\Phi_\alpha$$

Now fixing a q-simplex $\sigma \in \Delta[n-1]_q$, evaluation at σ gives point-wise lifts in the diagram

$$\begin{array}{ccccc} \partial\Delta[n] & \longrightarrow & \mathfrak{A}^p_\bullet \otimes C^q_\bullet & \xrightarrow{\mathrm{ev}_\sigma f} & \mathfrak{A}^p_\bullet \\ \downarrow & & & \nearrow & \\ \Delta[n] & & \cdots & & \end{array}$$

because $\mathfrak{A}^p_\bullet$ is a trivial Kan complex. That means that we have a map $\partial\Delta[n] \to \mathbf{Set}(\Delta[n-1]_q, \mathfrak{A}^p_n)$ which we can extend to the all q-simplices of the standard simplex $\Delta[n]$ in the same way as we have done in part 1).

The second part is an immediate consequence of the Künneth theorem over fields. □

The same proof as for $\mathcal{A}_{\mathrm{PL}}$ yields that both C^* and $\mathcal{A}_{\mathrm{PL}} \otimes C^*$ send injective maps to surjective maps.

Now we have the material to work on our main proof:

Theorem 4.7. *The maps $\mathcal{A}_{\mathrm{PL}} \to \mathcal{A}_{\mathrm{PL}} \otimes C^* \leftarrow C^*$ given by $a \mapsto a \otimes 1$ and $c \mapsto 1 \otimes c$ are quasi-isomorphisms of DGAs.*

Proof. The maps $\mathfrak{A}^*_\bullet \longrightarrow \mathfrak{A}^*_\bullet \otimes C^*_\bullet \longleftarrow C^*_\bullet$ are quasi-isomorphisms because all cohomology groups are R and both maps send 1 to $1 \otimes 1$, which is a generating representative.

Now we check that we have quasi-isomorphisms for every simplicial set. We only do this for the map $\mathcal{A}_{\mathrm{PL}} \to \mathcal{A}_{\mathrm{PL}} \otimes C^*$ because it is entirely the same for the other map.

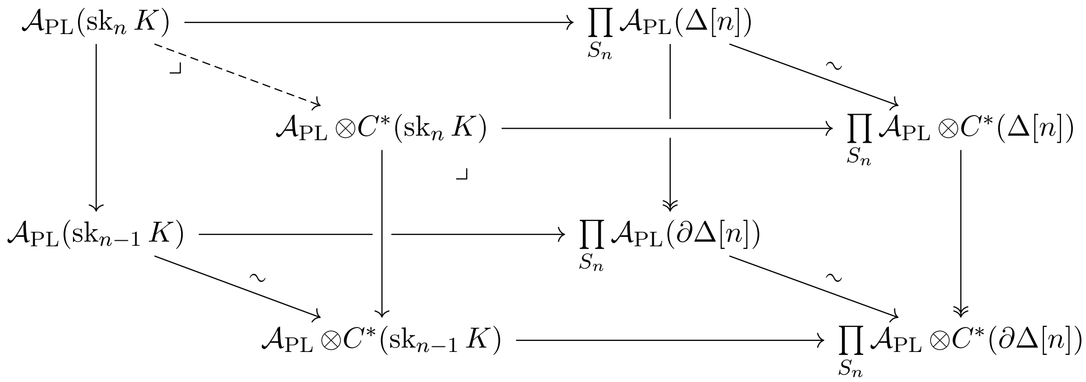
From corollary 2.9 we have the decomposition of a simplicial K set into a series of skeleta $\mathrm{sk}_n(K)$, such that $\mathrm{sk}_n(K)$ arises by attaching standard simplices $\Delta[n]$ to $\mathrm{sk}_{n-1}(K)$ and $K = \varinjlim \mathrm{sk}_n(K)$. First, consider we have already verified the claim for every n-skeleta $\mathrm{sk}_n(K)$. Because the inclusion $\mathrm{sk}_{n-1}(K) \to \mathrm{sk}_n(K)$ is injective (by theorem 2.8) and $\mathcal{A}_{\mathrm{PL}}$ and $\mathcal{A}_{\mathrm{PL}} \otimes C^*$ both send injective maps to surjective maps the inverse sequential systems $\mathcal{A}_{\mathrm{PL}}(\mathrm{sk}_n(K))$ and $C^*(\mathrm{sk}_n(K))$ are both Mittag-Leffler. It is a well known fact now that this implies that in this case cohomology commutes with limits. Therefore we have

$$\begin{aligned} H(\mathcal{A}_{\mathrm{PL}}(K)) =& H(\mathcal{A}_{\mathrm{PL}}(\varinjlim \mathrm{sk}_n K)) = H(\varprojlim \mathcal{A}_{\mathrm{PL}}(\mathrm{sk}_n K)) = \varprojlim H(\mathcal{A}_{\mathrm{PL}}(\mathrm{sk}_n K)) \\ &\cong \varprojlim H(\mathcal{A}_{\mathrm{PL}} \otimes C^*(\mathrm{sk}_n K)) = H(\varprojlim \mathcal{A}_{\mathrm{PL}} \otimes C^*(\mathrm{sk}_n K)) = H(\mathcal{A}_{\mathrm{PL}} \otimes C^*(K)) \end{aligned}$$

Now to proof the claim on the skeleta we use induction on n. For $n = -1$ the condition is vacuous. Let the statement already hold for every $(n-1)$-skeleton of every simplicial set. To make our induction step we look at the following pushout:

$$\begin{array}{ccc} \coprod_{S_n} \partial\Delta[n] & \longrightarrow & \mathrm{sk}_{n-1}(K) \\ \big\downarrow & & \big\downarrow \\ \coprod_{S_n} \Delta[n] & \longrightarrow & \mathrm{sk}_n(K) \end{array}$$

As $\mathcal{A}_{\mathrm{PL}}$ and C^* send colimits to limits and injective maps to surjective maps we get the following cube diagram:



with pullbacks at the front and back faces and the downwards arrows on the right surjections. The forward facing arrows are given by $\mathcal{A}_{\mathrm{PL}} \to \mathcal{A}_{\mathrm{PL}} \otimes C^*$ and because cohomology commutes with products the map

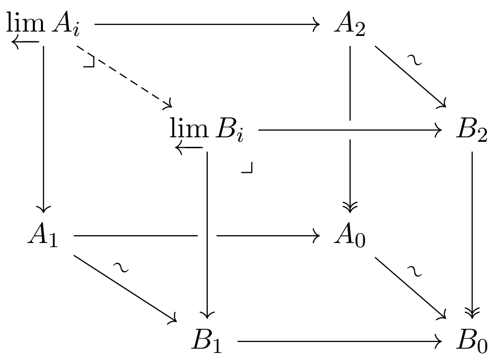
$$\prod_{S_n} \underbrace{\mathcal{A}_{\mathrm{PL}}(\Delta[n])}_{=\mathfrak{A}_n^*} \to \prod_{S_n} \underbrace{\mathcal{A}_{\mathrm{PL}} \otimes C^*(\Delta[n])}_{\mathfrak{A}_n^* \otimes C_n^*}$$

is a quasi-isomorphism by the first remark on the simplicial objects and the maps

$$\mathcal{A}_{\mathrm{PL}}(\mathrm{sk}_{n-1} K) \to \mathcal{A}_{\mathrm{PL}} \otimes C^*(\mathrm{sk}_{n-1} K), \qquad \prod_{S_n} \mathcal{A}_{\mathrm{PL}}(\partial\Delta[n]) \to \prod_{S_n} \mathcal{A}_{\mathrm{PL}} \otimes C^*(\partial\Delta[n])$$

are quasi-isomorphisms by induction hypothesis and because $\partial\Delta[n] = \mathrm{sk}_{n-1} \Delta[n]$. The statement now follows from the gluing lemma: □

Lemma 4.8 (Gluing Lemma)**.** *Consider the diagram in* **CDGA**



where the maps $A_2 \twoheadrightarrow A_0$ and $B_2 \twoheadrightarrow B_0$ are surjections and the maps $A_1 \to B_1$, $A_0 \to B_0$ and $A_2 \to B_2$ quasi-isomorphisms. Then the induced map between the pullbacks is a quasi-isomorphism as well.

Proof. We can write the pullbacks as the equalizers between the the two maps $A_1 \to A_0$ and $A_2 \to A_0$ resp. $B_1 \to B_0$ and $B_2 \to B_0$ after projection to the respective component:

$$A_1 \times A_2 \rightrightarrows A_0 \qquad\qquad B_1 \times B_2 \rightrightarrows B_0$$

In particular they are the kernel of corresponding difference map $A_1 \times A_2 \to A_0$ and $B_1 \times B_2 \to B_0$. Because $A_2 \twoheadrightarrow A_0$ and $B_2 \twoheadrightarrow B_0$ are surjcective the same is true for these difference maps. Thus we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim A_i & \longrightarrow & A_1 \times A_2 & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & \varprojlim B_i & \longrightarrow & B_1 \times B_2 & \longrightarrow & B_0 & \longrightarrow & 0 \end{array}$$

in which the two left downward arrows are still quasi-isomorphisms as cohomology commutes with products. The claim now follows form the 5-lemma applied to the long exact sequence in cohomology associated to this short exact sequence. □

Remark 4.9. The gluing lemma also holds in a more general setting of model categories with objects being fibrant objects and surjective and quasi-isomorphism replaced by fibration and weak equivalence (cf. [Rie14] Corollary 14.3.2.).

5. $\mathcal{A}_{\mathrm{PL}}$ and manifolds

The remark 2.5 suggests that there is a strong link between the piecewise linear differential forms and the de Rham differential forms. In this section we will show that for a smooth manifold M we actually have a zigzag of quasi-isomorphisms

$$\mathcal{A}_{\mathrm{PL}}(M, \mathbb{R}) \longrightarrow \bullet \longleftarrow \Omega^*(M)$$

between the real piecewise differential forms on M and the de Rham differential forms on M. Therefore in this section all coefficients are in $\mathbb{R}$ and all polynomial and all piecewise differential forms are over $\mathbb{R}$. Besides from working over $\mathbb{R}$ we need to consider another subtlety: In the category of smooth manifolds morphisms are smooth maps, but until now everything has only been continuous. Thus we need:

Definition 5.1. Let M be a smooth manifold. The *set of smooth simplices* of M is the subset $\mathrm{Sing}^\infty(M) \subset \mathrm{Sing}(M)$ consisting of all maps $|\Delta_n| \to M$, that can be extended to smooth functions from an open neighborhood of $|\Delta_n|$.

From the inclusion $\mathrm{Sing}^\infty(M) \to \mathrm{Sing}(M)$ we now get a map

$$\mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}\, M, \mathbb{R}) \to \mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}^\infty(M), \mathbb{R})$$

which will turn out to be a quasi-isomorphism. Our next ingredient will be to do a similar trick as in section 4 and find a "good" sCDGA $\Omega^*_\bullet \supset \mathfrak{A}^*_\bullet$ such that for $\Omega^* := \mathrm{Lan}_Y \Omega^*_\bullet$: **sSet** $\to$ **CDGA** there is a quasi-isomorphism:

$$\Omega^*(M) \to \Omega^*(\mathrm{Sing}^\infty(M)).$$

We define $\Omega^*_\bullet$ by

$$\Omega^*_n = \Omega^*(|\Delta_n|)$$

where a smooth differential form on $|\Delta_n|$ is a restriction of a smooth differential form on an open neighborhood containing $|\Delta^n|$. It is clear by 2.5 that $\mathfrak{A}^*_\bullet \subset \Omega^*_\bullet$ and under the functor $\mathrm{Lan}_Y(-)$ this becomes a map

$$\mathcal{A}_{\mathrm{PL}}(K) \to \Omega^*(K)$$

for any simplicial set K, in particular for $\mathrm{Sing}^\infty(M)$. Note that in some sense we restrict Ω^* along the geometric realization functor but left Kan extend $\Omega^*_\bullet$ along the Yoneda embedding and apply Sing^∞. Therefore it is not clear that we get our map $\Omega^* M \to \Omega^*(\mathrm{Sing}^\infty(M))$. By precise analysis of the involving functors, especially by defining the smooth realization functor properly, we could actually get this map from the universal property of pointwise Kan extensions. But we will just give an explicit formula. Looking at the formula for left Kan extensions yields:

$$\Omega^*(\mathrm{Sing}^\infty(M)) = \left(\int^n \mathbf{sSet}(\mathrm{Sing}^\infty(M), \Delta[n]) \cdot \Omega^*_n\right)^{op} = \mathbf{sSet}(\mathrm{Sing}^\infty M, \Omega^*_\bullet)$$

and we can define the map CDGA map

$$\Omega^* M \to \Omega^*(\mathrm{Sing}^\infty M); \quad \omega \mapsto (\sigma \mapsto \sigma^*\omega)$$

Now summing up what we have, we get for every smooth manifold M a chain of maps:

$$\mathcal{A}_{\mathrm{PL}}(M) \to \mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}^\infty(M)) \to \Omega^*(\mathrm{Sing}^\infty(M)) \leftarrow \Omega^*(M)$$

and it remains to show that these are indeed quasi-isomorphisms. We will need the following two statements:

Lemma 5.2. *The underlying simplicial set of $\Omega^p_\bullet$ is a trivial Kan complex for every p.*

Proof. Given a map $\partial\Delta[n] \to \Omega^p_\bullet$ let $\omega_i \in \Gamma(|\Delta^{n-1}|, \Lambda^p(T^*|\Delta^{n-1}|))$ denote the images of the non-degenerate $(n-1)$-simplices. Similarly to the case $C^p_\bullet$ choose $\omega'_i \in \Gamma(|d_i\Delta^{n-1}|, \Lambda^p(T^*(|\Delta_n|))$ such that for $s \in |\Delta^{n-1}|, \xi_1, \ldots, \xi_p \in T_{d_i s}|\Delta^{n-1}|$

$$\omega'_i(d_i s)(d_{i*}\xi_1, \ldots, d_{i*}\xi_p) = \omega_i(s)(\xi_1, \ldots, \xi_p)$$

The same calculation as before, using $d_i\omega_j = d_{j-1}\omega_i$ for $i < j$, gives us

$$\omega'_j d_j d_i(s)(d_{j*}d_{i*}\xi_1, \ldots, d_{j*}d_{i*}\xi_p) = \omega'_i d_j d_i(s)(d_{j*}d_{i*}\xi_1, \ldots, d_{j*}d_{i*}\xi_p)$$

for all $i < j$ and the maps coincide in $|d_j\Delta^{n-1}| \cap |d_i\Delta^{n-1}|$. Thus we have a section $\bigcup |d_i\Delta^{n-1}| \to \Lambda^p T^*(|\Delta_n|)$ which we can smoothly extend to entire $|\Delta_n|$. □

Lemma 5.3 (Bootstrap Lemma). *For a smooth manifold M a statement $P(U)$ for all open subsets $U \subset M$ is true if and only if it is true in these three cases:*

1) $P(U)$ is true for U diffeomorphic to a convex subset of $\mathbb{R}^n$.

2) If for $P(U), P(V), P(U \cap V)$ is true for $U, V \subset M$ open, also is $P(U \cup V)$.

3) If $P(U_\alpha)$ for disjoint $\{U_\alpha\}$, then $P\left(\bigcup U_\alpha\right)$.

A proof of this useful lemma can be found in [Bre93] lemma V.9.5. In particular in the case of the above lemma the statement is true for M itself. That is how we are going to deal with our outer maps from above:

Proposition 5.4. *The map $\mathcal{A}_{\mathrm{PL}}(\operatorname{Sing} M) \to \mathcal{A}_{\mathrm{PL}}(\operatorname{Sing}^\infty M)$ is a quasi-isomorphism.*

Proof. By theorem 4.7 it is enough to show that $C^*(\operatorname{Sing} M) \to C^*(\operatorname{Sing}^\infty(M))$ is a quasi-isomorphism. By UCT it is sufficient to show it on the chain complex C_*. Let $U \subset M$ be diffeomorphic to a convex open subset of $\mathbb{R}^n$. It can be shown (e.g. [Bre93] in theorem IV.15.5) that the map

$$D : C_*U \to C_{*+1}U;\ (D\sigma)\left(\sum_{i=1}^n \lambda_i e_i\right) = \lambda_0 \sigma\left(\sum_{i=1}^n \frac{\lambda_i}{1-\lambda_0} e_i\right)$$

is a chain homotopy between the identity and the map $\epsilon : C_*X \to C_*X$ zero in positive degrees and identity in C_0X. Because all operations in D are smooth, if σ is a smooth simplex, the same is true for $D(\sigma)$. That means D restricts to a map

$$D^\infty : C_*(\operatorname{Sing}^\infty U) \to C_{*+1}(\operatorname{Sing}^\infty U)$$

which is by the same calculations a chain contraction. Therefore the map $C_*(U) \to C_*(\operatorname{Sing}^\infty U)$ is a quasi-isomorphism.

Now let $U, V \subset M$ open, such that the statement is true for $U \cap V$, U and V. A quick look at the barycentric subdivision map and the associated chain homotopy ([Bre93] IV.17) reveals that they restrict to smooth simplices as well. Therefore the proof that for a cover $\mathcal{U}$ of a space X the inclusion $\operatorname{Sing}_{\mathcal{U}} X \hookrightarrow \operatorname{Sing} X$ of $\mathcal{U}$-small simplices into $\operatorname{Sing} X$ induces an quasi-isomorphis $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ also shows that $C_*(\operatorname{Sing}_{\mathcal{U}}^\infty(X)) \hookrightarrow C_*(\operatorname{Sing}^\infty X)$ is a quasi-isomorphism for $\operatorname{Sing}_{\mathcal{U}}^\infty = \operatorname{Sing}^\infty \cap \operatorname{Sing}_{\mathcal{U}}$. But now we get short exact sequences for $\mathcal{U} = \{U, V\}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*^{\mathcal{U}}(U \cap V) & \longrightarrow & C_*^{\mathcal{U}}(U) \oplus C_*^{\mathcal{U}}(V) & \longrightarrow & C_*^{\mathcal{U}}(U \cup V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_*(\operatorname{Sing}_{\mathcal{U}}^\infty(U \cap V)) & \to & C_*(\operatorname{Sing}_{\mathcal{U}}^\infty U) \oplus C_*(\operatorname{Sing}_{\mathcal{U}}^\infty V) & \to & C_*(\operatorname{Sing}_{\mathcal{U}}^\infty(U \cup V)) & \to & 0
\end{array}$$

After applying the snake lemma, 5-lemma and these above quasi-isomorphism from subdivision we get that $C_*(U \cup V) \to C_*(\operatorname{Sing}^\infty(U \cup V))$ is a quasi-isomorphism.

Finally let $\{U_\alpha\}$ be a collection of disjoint subsets of M for which the statement is true. Obviously

$$C_*\left(\coprod U_\alpha\right) = \bigoplus C_*(U_\alpha) \simeq \bigoplus C_*(\operatorname{Sing}^\infty U_\alpha) = C_*\left(\operatorname{Sing}^\infty\left(\coprod U_\alpha\right)\right)$$

And the claim follows from the bootstrap lemma. □

Proposition 5.5. *The map* $\mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}^\infty M) \to \Omega^*(\mathrm{Sing}^\infty M)$ *is a quasi-isomorphism.*

Proof. This statement is even correct for any simplicial set instead of $\mathrm{Sing}^\infty M$. We have seen that $\mathfrak{A}^*_\bullet$ and $\Omega^*_\bullet$ have trivial Kan complexes as underlying simplicial sets. Moreover, as $|\Delta_n| \subset \mathbb{R}^{n+1}$ is contractible, the Poincaré lemma A.3 yields

$$H\Omega^*_n = H(\Omega^*(|\Delta_n|) = \mathbb{R}.$$

Because also $H\mathfrak{A}^*_n = \mathbb{R}$ the inclusion $\mathfrak{A}^*_\bullet \hookrightarrow \Omega^*_\bullet$ has to be a quasi-isomorphism. The proof on theorem 4.7 by induction on the skeleta of simplicial sets gives us the quasi-isomorphism for any simplicial set. □

Proposition 5.6. *The map* $\Omega^*(M) \to \Omega^*(\mathrm{Sing}^\infty(M))$ *is a quasi-isomorphism.*

Proof. Let $U \subset M$ be diffeomorphic to a convex subset of $\mathbb{R}^n$. By the Poincaré lemma (lemma A.3) we have $H(\Omega^* U) = \mathbb{R}$ and by all the above we know

$$H(\Omega^*(\mathrm{Sing}^\infty U)) = H(\mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}^\infty U)) = H(C^* U) = \mathbb{R}.$$

Now $U, V \subset M$ are open subsets, such that we have quasi-isomorphisms on $U \cap V, U$ and V. Recall the formular for the piecewise linear de Rham forms $\Omega^*(\mathrm{Sing}^\infty(M)) = \mathbf{sSet}(\mathrm{Sing}^\infty M, \Omega^*_\bullet)$. So by choosing a refinement of $\mathrm{Sing}^\infty_{\mathcal{U}} \hookrightarrow \mathrm{Sing}^\infty$ subordinate to the cover $\mathcal{U} = \{U, V\}$ the bottom row in the following diagram is exact (injectivity on the left is the only non-trivial part):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^*(U \cup V) & \longrightarrow & \Omega^*(U) \oplus \Omega^*(V) & \longrightarrow & \Omega^*(U \cap V) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \Omega^*(\mathrm{Sing}^\infty(U \cup V)) & \to & \Omega^*(\mathrm{Sing}^\infty U) \oplus \Omega^*(\mathrm{Sing}^\infty V) & \to & \Omega^*(\mathrm{Sing}^\infty(U \cap V)) & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^*(\mathrm{Sing}^\infty_{\mathcal{U}}(U \cup V)) & \to & \Omega^*(\mathrm{Sing}^\infty_{\mathcal{U}} U) \oplus \Omega^*(\mathrm{Sing}^\infty_{\mathcal{U}} V) & \to & \Omega^*(\mathrm{Sing}^\infty_{\mathcal{U}}(U \cap V)) & \to & 0
\end{array} \tag{5.1}$$

By lemma A.4 the first row is also exact and we would like to have quasi-isomorphisms between the rows. In the proposition 5.4 above we have already used that the inclusion $\mathrm{Sing}^\infty_{\mathcal{U}} \hookrightarrow \mathrm{Sing}^\infty$ induces a quasi-isomorphism between the chain complexes. By UCT applied to this map and by proposition 5.5 and theorem 4.7 we get a diagram

$$\begin{array}{ccccc}
C^*(\mathrm{Sing}^\infty X) & \simeq & \mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}^\infty X) & \xrightarrow{\sim} & \Omega^*(\mathrm{Sing}^\infty X) \\
\downarrow \wr & & \downarrow & & \downarrow \\
C^*(\mathrm{Sing}^\infty_{\mathcal{U}} X) & \simeq & \mathcal{A}_{\mathrm{PL}}(\mathrm{Sing}^\infty_{\mathcal{U}} X) & \xrightarrow{\sim} & \Omega^*(\mathrm{Sing}^\infty_{\mathcal{U}} X)
\end{array}$$

with all horizontal and the left vertical maps quasi-isomorphisms. It follows that the other vertical maps are quasi-isomorphisms as well. That means in the diagram (5.1) all vertical maps on the bottom are quasi-isomorphisms. By assumption on U and V also the two right vertical maps on the top. The long exact sequence argument shows that $\Omega^*(U\cup V)\to\Omega^*(\mathrm{Sing}^\infty_{\mathcal{U}}(U\cup V))$ is a quasi-isomorphism and finally the 2-out-of-3 property for quasi-isomorphisms proves the statement for $U\cup V$.

For the last step let $\{U_\alpha\}$ be a collection of disjoint open sets of M, such that the statement is true for every U_α. Then we have

$$\Omega^*\left(\coprod U_\alpha\right)=\prod\Omega^*(U_\alpha)\simeq\prod\Omega^*(\mathrm{Sing}^\infty U_\alpha)=\Omega^*\left(\mathrm{Sing}^\infty\left(\coprod U_\alpha\right)\right)$$

because cohomology commutes with products and Sing^∞ and both Ω^* commute with disjoint unions. Now the bootstrap lemma finishes the proof. □

We have finally proven:

Theorem 5.7. *For every manifold M there is a zigzag of quasi-isomorphisms of CDGAs from $\mathcal{A}_{\mathrm{PL}}(M,\mathbb{R})$ to $\Omega^*(M)$.* □

6. Formal Spaces

An application of the functor $\mathcal{A}_{\mathrm{PL}}$ is the notion of formal spaces. Making precise the idea that a space has not too complicated piecewise linear differential forms. Note that for a pointed space (X,x) the inclusion $\{x\}\to X$ gives an augmentation $\mathcal{A}_{\mathrm{PL}}X\to\mathbb{Q}$. Then we define:

Definition 6.1. A CDGA A^* is called formal if there is a zigzag of quasi-isomorphisms of CDGAs

$$A^*\longrightarrow\bullet\longleftarrow H(A^*)$$

with the cohomology interpreted as a CDGA with zero differential. An augmented CDGA A^* is called *augmented formal*, if there is such a zigzag of quasi-isomorphisms of augmented CDGAs.

A space X is formal if the rational $\mathcal{A}_{\mathrm{PL}}X$ is a formal CDGA and a pointed space X is augmented formal if the rational $\mathcal{A}_{\mathrm{PL}}X$ is augmented formal.

Obviously every augmented formal space is formal, but the other direction is not clear and needs more sophisticated arguments about CDGAs. But first we can give some nice consequences of this definition:

Proposition 6.2. *Let X and Y be formal spaces, such that one is homologically finite. Then $X\times Y$ is formal.*

Proof. Let $p_1:X\times Y\to X$ and $p_2:X\times Y\to Y$ be the natural projections and μ the multiplication of $\mathcal{A}_{\mathrm{PL}}(X\times Y)$, then we want to show that

$$\mathcal{A}_{\mathrm{PL}}X\otimes\mathcal{A}_{\mathrm{PL}}Y\xrightarrow{\mathcal{A}_{\mathrm{PL}}(p_1)\otimes\mathcal{A}_{\mathrm{PL}}(p_2)}\mathcal{A}_{\mathrm{PL}}(X\times Y)\otimes\mathcal{A}_{\mathrm{PL}}(X\times Y)\xrightarrow{\mu}\mathcal{A}_{\mathrm{PL}}(X\times Y)$$

is a quasi-isomorphism of CDGAs. First of all this is indeed a CDGA map because as induced maps $\mathcal{A}_{\mathrm{PL}}(p_1)$ and $\mathcal{A}_{\mathrm{PL}}(p_2)$ are and the multiplication is by remark 1.3. Now applying cohomology we get a map

$$H^*X\otimes H^*Y\to H(\mathcal{A}_{\mathrm{PL}}X\otimes\mathcal{A}_{\mathrm{PL}}Y)\to H^*(X\times Y);\qquad f\otimes g\mapsto p_1^*f\cup p_2^*g \tag{6.1}$$

and some cup product calculus gives $p_1^* f \cup p_2^* g = f \times 1 \cup 1 \times g = f \times g$. That means the map (6.1) is just the cohomology cross product, which is an isomorphism by the Künneth theorem if one space is homologically finite. From the algebraic Künneth theorem we also get that if $\mathcal{A}_{\mathrm{PL}} X$ and $\mathcal{A}_{\mathrm{PL}} Y$ are formal $\mathcal{A}_{\mathrm{PL}} X \otimes \mathcal{A}_{\mathrm{PL}} Y$ is formal as well. □

Proposition 6.3. *Let X, Y be pointed and augmented formal spaces. Then $X \vee Y$ is augmented formal.*

Proof. We first show that the natural map $\mathcal{A}_{\mathrm{PL}}(X \vee Y) \to \mathcal{A}_{\mathrm{PL}} X \times_{\mathbb{Q}} \mathcal{A}_{\mathrm{PL}} Y$ from the piecewise linear differential forms on $X \vee Y$ to the fibered product over $\mathbb{Q}$ of the piecewise linear differential forms on X and Y induced by the universal property of the fiber product is a quasi-isomorphism. From the explicit description of the abelian groups of $\mathcal{A}_{\mathrm{PL}} X \times_{\mathbb{Q}} \mathcal{A}_{\mathrm{PL}} Y$ as the direct sums in positive degree and the direct sum modulo $\mathbb{Q}$ in degree zero, we can calculate that in this concrete case cohomology commutes with the fiber product. Therefore we get after applying cohomology

$$H^*(X \vee Y) \to H^* X \times_{\mathbb{Q}} H^* Y,$$

the induced map of the fiber product in cohomology which is an isomorphism. Finally the fiber product over $\mathbb{Q}$ of augmented formal CDGAs is again augmented formal by an iterated application of the gluing lemma 4.8. □

At this point we will give some examples of formal spaces. In order to even use some of the things about the de Rham complex in combination with the piecewise-linear differential forms we use the following fact, to which there is a reference in [FHT01] section 12.

Theorem 6.4. *Let X be a topological space. If $\mathcal{A}_{\mathrm{PL}}(X, R)$ is formal for any field extension $R/\mathbb{Q}$, then X is formal.*

In particular by theorem 5.7 a manifold is formal if $\Omega^* M$ is formal. Therefore in this case we have two ways to see if they are formal:

Example 6.5. Spheres are formal. We will give both proofs:

1) We can find our quasi-isomorphism directly from $\mathcal{A}_{\mathrm{PL}} S^n$. First let n be odd. Then

$$(H^*(S^n), 0) \to \mathcal{A}_{\mathrm{PL}}(S^n)$$

sending the generator of $H^*(S^n)$ to a representative x of its cohomology class in $\mathcal{A}_{\mathrm{PL}}$. Because n is odd $x^2 = 0$ and the map is a CDGA map.

If n is even we will only find a zigzag: Because there is no representative x of the generating cohomology class that squares to zero, we need an element y in degree $2n - 1$ with $dy = x^2$ to kill all higher cohomology. But now

$$(\Lambda(x, y), d) \to \mathcal{A}_{\mathrm{PL}}$$

is a quasi-isomorphism. Sending x to the generator of $H^n(S^n)$ and y to 0 gives another quasi-isomorphism:

$$(H^*(S^n), 0) \leftarrow (\Lambda(x, y), d) \to \mathcal{A}_{\mathrm{PL}}(S^n)$$

2) Or we can use the results from last chapter. Consider the quasi-isomorphisms

$$\mathcal{A}_{\mathrm{PL}}(S^n, \mathbb{R}) \simeq \Omega^*(S^n).$$

Now S^n has only cohomology in top dimension and it follows that every representative of a cohomology class in $\Omega^n(S^n)$ squares to zero because $\Omega^{2n}(S^n) = 0$. Choosing one representative yields the desired quasi-isomorphism

$$H^*(S^n) \to \Omega^*(S^n) \simeq \mathcal{A}_{\mathrm{PL}}(S^n, \mathbb{R})$$

Remark 6.6. By looking at the involved CDGAs it follows that all maps can be extended to maps of augmented CDGAs. That is spheres are even augmented formal. By the propositions 6.2 and 6.3 we have that products and wedges of spheres are also formal.

Appendices

In the appendix we want to give some results on de Rham cohomology and Steenrod operations. Both constructions are not directly connected to piecewise linear differential forms, but still fit into a broader context. Throughout this paper we have used several facts of the de Rham cohomology, so in appendix A we will prove these statements. The appendix B elaborates on the commutativity problem mentioned in the preface that points out the distinct property of the piecewise linear differential forms to be a commutative DGA.

A. De Rham Cohomology

We need some tools concerning the de Rham cohomology of a manifold. We will start here from the beginning:

Definition A.1. Let M be a smooth manifold. Then define the graded de Rham algebra $\Omega^*(M)$ by

$$\Omega^p(M) = \Gamma(\Lambda^p T^* M)$$

as the sections of the pth exterior algebra of the cotangent bundle over M with the graded commutative multiplication map $\wedge$ inherited from $\Lambda^*(T^*M)$.

For $x \in M$ let dx_i denote the dual of the standard basis of T_xM. Then $\{dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p} : I = (i_1 < \cdots < i_p) \subset \{1, \ldots, n\}\}$ is a basis of $\Lambda^p(T^*_xM)$ and every $\omega \in \Omega^p$ is of the form

$$\omega = \sum_I f_I dx_I$$

for $f \in C^\infty(M)$. Now we can make $\Omega^*(M)$ into a CDGA by defining a differential on generators:

$$d : \Omega^p(M) \to \Omega^{p+1}(M); \quad d(f dx_I) = df \wedge dx_I$$

with $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. It follows directly from $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$ that $d^2 = 0$ and this is indeed a differential. Moreover d is independent of the choice of the basis in $T_x^* M$ (cf. [MT97] theorem 3.7). Thus we have:

The constructions are natural and Ω^* defines a functor $\mathbf{Man}^\infty \to \mathbf{CDGA}^{op}$ from smooth manifolds to CDGAs. For a map $f : M \to N$ between smooth manifolds we get a pullback $f^* := \Omega^*(f) : \Omega^*(N) \to \Omega^*(M)$ via

$$f^* \omega(x)(\xi_1, \ldots, \xi_p) = \omega(f(x))(f_* \xi_1, \ldots, f_* \xi)$$

for $\omega \in \Omega^p M, x \in M$ and $\xi_1, \ldots, \xi_p \in T_x M$. Straight forward calculations show that this is indeed a CDGA map (cf. [MT97] theorem 3.12).

Remark A.2. From the definition we get that $\Omega^p(M) = 0$ for all $p > \dim M$. In particular $\Omega^*(pt) = C^\infty(pt) = \mathbb{R}$.

This yields a canonical augmentation $\epsilon : \Omega^*(M) \to \mathbb{R}$ for every point $x \in M$ given by the pullback of the constant map $M \to \{x\}$. That is $\epsilon(\omega) = \omega(x)$ in degree 0 and $\epsilon(\omega) = 0$ otherwise.

Lemma A.3 (Poincaré-Lemma)**.** *If $U \subset \mathbb{R}$ is a convex open subset, then $H(\Omega^*(U)) = \mathbb{R}$.*

Proof. We will show that the unit $\eta : \mathbb{R} \to \Omega^*(U)$ and the augmentation ϵ for any point $x \in U$ are chain homotopy inverses. The first direction is clear. Thus we have to find a chain homotopy for $\mathrm{id} \simeq \eta\epsilon$. Note that all these three maps are pullbacks of $\mathrm{id} : U \to U$ resp. $\{x\} \to M$ resp. $U \to \{x\}$ and that $\mathrm{id} \simeq (U \to \{x\} \to U)$ by assumption. We will now give a general proof, that homotopic maps between spaces give rise to chain homotopic pullback maps between the de Rham complexes. The claim then follows.

The two maps ϕ_ν^* with $\phi_\nu : U \to U \times \mathbb{R}; x \mapsto (x, \nu)$ for $\nu = 0, 1$ are chain homotopic. The chain homotopy is given by

$$S_p : \Omega^p(U \times \mathbb{R}) \to \Omega^p(U); \omega = \sum f_I dx_I + \sum g_J dt \wedge dx_J \mapsto \sum \left(\int_0^1 g_J(x,t) dt \right) dx_J$$

because indeed

$$\begin{aligned} dS_p\omega + S_{p+1}d\omega &= \sum_{J,i} \left(\int_0^1 \frac{\partial g_J(x,t)}{\partial x_i} dt \right) dx_i \wedge dx_J \\ &+ \left(\sum_I \left(\int_0^1 \frac{\partial f_I(x,t)}{\partial t} dt \right) dx_I - \sum_{J,i} \left(\int_0^1 \frac{\partial g_J(x,t)}{\partial x_i} dt \right) dx_i \wedge dx_J \right) \\ &= \sum_I f_I(x,1) dx_I - \sum f_I(x,0) dx_I = \phi_1^* \omega - \phi_0^* \omega \end{aligned}$$

due to $\phi_\nu^*(dt) = 0$. Now given a contraction $D : U \times \mathbb{R} \to U$ of U to the point x we have

$$D \circ \phi_0 = \mathrm{id} : U \to U, \qquad D \circ \phi_1 = U \to \{p\} \to U : U \to U.$$

And it follows immediately that $S \circ D^*$ is a chain homotopy between $D \circ \phi_0$ and $D \circ \phi_1$. $\square$

Lemma A.4. *For $U, V \subset M$ open with i_U, i_V the obvious inclusions into $U \cap V$ and j_U, j_V the obvious inclusion from $U \cap V$ the short sequence*

$$0 \longrightarrow \Omega^*(U \cup V) \xrightarrow{(i_U^*, i_V^*)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_U^* - j_V^*} \Omega^*(U \cap V) \longrightarrow 0$$

is exact.

Proof. Everything is straightforward except for surjectivity on the right. Let $\omega \in \Omega^p(U \cap V)$. Choose a partition of unity on $U \cup V$ subordinate to U, V. Summing up all functions with support in U we get a function $f : U \cup V \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & x \in U \setminus V \\ 0 & x \in V \setminus U \end{cases}$$

Now we can extend $-f\omega$ with 0 to a differential form on entire V and $(1 - f)\omega$ with 0 to a differential form on entire U and get our element in $\Omega^p(U) \oplus \Omega^p(V)$. □

By the snake lemma we now get a Mayer-Vietoris sequence for the de Rham cohomology natural in U and V.

B. Commutativity Problem

The piecewise polynomial differential forms are very interesting because they yield a commutative model of the cochain algebra. This relays very strongly on the fact, that our coefficients contained the field $\mathbb{Q}$.

In this section we will proof that for example with coefficients in $\mathbb{F}_2$ there cannot be a commutative model of C^*. We will use the Steenrod squares which are cohomology operations defined via the non-commutativity of any multiplication on the cochains over $\mathbb{F}_2$.

B.1. Cohomology Operations and Eilenberg-MacLane-Spaces

Definition B.1. An *Eilenberg-MacLane-space* $K(\pi, n)$ for a natural number n and a group π is a topological space satisfying

$$\pi_k(K(\pi, n)) = \begin{cases} \pi & k = n \\ 0 & \text{else} \end{cases}$$

We have a couple of facts about these spaces, which can be found in every standard textbook on topology, e.g. [Bre93]:

Theorem B.2. *For every group π there exists a space $K(\pi, 1)$ and for every abelian group π and any $n \in \mathbb{N}$ there exists a $K(\pi, n)$. Moreover, for fixed π and n they are unique up to weak homotopy equivalence and can be chosen to be CW-complexes - even of finite type if the group is finitely generated.*

Obviously for $n \geq 2$ we need π to be abelian because all higher homotopy groups are. The concrete construction can be found in [Bre93] corollary VII.11.9.

Example B.3. We have $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$ because the universal covering $S^\infty \to \mathbb{R}P^\infty$ is a double covering yielding $\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2$ and $\pi_n(\mathbb{R}P^\infty) = \pi_n(S^\infty) = 0$ otherwise.

One can see in this example that Eilenberg-MacLane-spaces can have complicated cohomology. In this cases $H^*(\mathbb{R}P^\infty, \mathbb{F}_2) = \mathbb{F}_2[\alpha]$, with $|\alpha| = 1$.

We can, however, choose a canonical element of $H^n(K(\pi, n), \pi)$. Because we have $\pi_k(K(\pi, n)) = 0$ for $k < n$ the Hurewicz map $h : \pi_n(K(\pi, n)) = \pi \to H_n(K(\pi, n))$ is an isomorphism. Now $\iota_n := [h^{-1}] \in H^n(K(\pi, n), \pi)$ is our wanted cohomology class.

An important property of these spaces is the following theorem:

Theorem B.4. *The functor $H^n(-, \pi)$ is represented by $K(\pi, n)$, i.e. for a CW complex X we have a natural isomorphism*

$$[X, K(\pi, n)] \to H^n(X, \pi)$$

between homotopy classes of maps $X \to K(\pi, n)$ and cohomology classes of $H^n(X, \pi)$.

This is proved in detail in section VII.12 in [Bre93]. The proof yields furthermore that the element ι_n is the image of the identity on $K(\pi, n)$.

Having looked at the cohomology functor itself, we can also look at its natural transformations:

Definition B.5. A *cohomology operation* θ of type $(n, \pi; k, \omega)$ with $n, k \in \mathbb{N}$ and $\pi, \omega \in$ **Ab** is a natural transformation

$$\theta : H^n(-, \pi) \to H^k(-, \omega)$$

We will denote the set of all cohomology operations of type $(n, \pi; k, \omega)$ by $\mathcal{O}(n, \pi; k, \omega)$. A cohomology operation is called *stable* if it commutes with suspensions.

Now as a direct consequence of the Yoneda lemma we obtain:

Theorem B.6. *We have a bijection*

$$\mathcal{O}(n, \pi; k, \omega) \to H^k(K(\pi, n), \omega)$$

given by $\theta \mapsto \theta(\iota_n)$. □

Now we have enough general material about cohomology operations and we can start to explicitly construct the Steenrod operations:

B.2. Steenrod Squares

From now on we work in $\mathbb{F}_2$-coefficients. Let

$$\Delta_0 : C_*X \to C_*X \otimes C_*X$$

be a diagonal approximation of the chain complex of a space X and denote by τ the flip map

$$\tau : C_*X \otimes C_*X \to C_*X \otimes C_*X; \quad a \otimes b \to b \otimes a$$

By an acyclic models argument any diagonal approximations are chain homotopic. Now Δ_0 and $\tau \circ \Delta_0$ are both diagonal approximations, such that there is a chain homotopy $\Delta_1 : C_*X \to (C_*X \otimes C_*X)_{*+1}$ between them with $\Delta_1 = 0$ on C_0X.

It follows that the map $(1+\tau)\Delta_1$ is a chain map of degree 1 with $(1+\tau)\Delta_1 = 0$ on C_0X because

$$\partial(1+\tau)\Delta_1 + (1+\tau)\Delta_1\partial = (1+\tau)(\partial\Delta_1 + \Delta_1\partial) = (1+\tau)(1+\tau)\Delta_0 = 0.$$

The same is true for the degree 1 zero map. Again by acyclic models we find a chain homotopy Δ_2 between $(1+\tau)\Delta_1$ and 0. Continuing in this manner we get a sequence $(\Delta_i)_{i\in\mathbb{N}}$ of chain maps $C_*X \to C_*X \otimes C_*X$ of degree i such that

$$(1+\tau)\Delta_i = \partial\Delta_{i+1} + \Delta_{i+1}\partial$$

measuring how non-commutative our diagonal approximation Δ_0 is, i.e. $\Delta_i = 0$ for $i > 0$ if Δ_0 is commutative. Indeed, this is the point we want to get our hand on. But first we also get these maps on the cochain complex:

$$\cup_i : C^*X \otimes C^*X \to (C_*X \to C_*X)^\vee \xrightarrow{\Delta_i^\vee} C^*X$$

which we call the $\cup_i$*-product*. It sends an p and q cocycle to an $p+n-i$ cocycle.

Now we will need a technical lemma involving the $\cup_i$-product ([Bre93] VI.16.1/2):

Lemma B.7. *We have the following to identities*

1) $da \cup_{i+1} da = d(a \cup_{i+1} da) + d(a \cup_i a)$
2) $(a+b) \cup_{i+1} (a+b) = a \cup_i a + b \cup_i b + da \cup_{i+1} b + a \cup_{i+1} db + d(a \cup_{i+1} b)$

Finally we have all the material to actually define the Steenrod operations that we are interested in:

Definition B.8. The *i-th Steenrod square* $Sq^i : H^nX \to H^{n+i}X$ is defined as $[a] \mapsto [a \cup_{n-i} a]$.

Juggling around with the last lemma (or ring [Bre93] theorem VI.16.3), one sees that Sq^i is well defined. Now there are two things to say:

Proposition B.9. *Sq^i is independent of the choice of a $\cup_i$-product, i.e. independent of the choice of chain homotopies Δ_i.*

This basically follows from acyclic models. Because any other choice of chain homotopies would lead to higher chain homotopies between them. That implies that these choices yield the same map after taking cohomology. In detail [Bre93] VI.16.5.

Theorem B.10. *Sq^i is a stable cohomology operation of type $(n, \mathbb{Z}/2; n+i, \mathbb{Z}/2)$. In particular it commutes with suspensions.*

Proof. Sq^i is natural, therefore a cohomology operation. To see that it commutes with suspension we look at the sequence

$$\tilde{H}^* X \xrightarrow{\delta} H^{*+1}(CX, X) \xleftarrow{\simeq} H^{*+1}(\Sigma X)$$

with the δ the boundary operator of the long exact sequence of the pair (CX, X) and the isomorphism $\tilde{H}^*(X, A) \cong \tilde{H}^*(X/A)$. Thus it suffices to prove that Sq^i commutes with boundary operators:

$$\begin{array}{ccc} H^q A & \xrightarrow{\delta} & H^{q+1}(X, A) \\ \downarrow{\scriptstyle Sq^j} & & \downarrow{\scriptstyle Sq^j} \\ H^{q+i} A & \xrightarrow{\delta} & H^{q+i+1}(X, A) \end{array}$$

Now let $[a] \in H^q A$, $[b] \in H^q X$ and $[c] \in H^{q+1}(X, A)$ such that $j^* c = db$ and $i^* b = a$ for the inclusions $i : A \to X$, and $j : X \to (X, A)$, i.e. $\delta[a] = [c]$. Then a calculation shows similar results for the $\cup_i$-products with $i = q - j$:

$$j^*(c \cup_{i+1} c) = d(b \cup_{i+1} db + b \cup_i b) \qquad i^*(b \cup_{i+1} db + b \cup_i b) = a \cup_i a$$

That means

$$Sq^j \delta[a] = Sq^j [c] = [c \cup_{i+1} c] = \delta[a \cup_i a] = \delta Sq^j [a]$$ □

So far we have no explicit way to calculate any steenrod square other than $Sq^i(a) = a^2$ for $a \in H^i X$. The purpose of this section is to show that there is no natural commutative structure on $C^* X$. If we can show that there is an Sq^i that is non-trivial on $H^j X$ for $i \neq j$ then we have proven that every diagonal approximation is not commutative. Indeed we have:

Theorem B.11. *Sq^0 is the identity.*

Proof. Sq^0 and $\mathrm{id}_{H^* X}$ are both cohomology operations of type $(n, \mathbb{Z}/2; n, \mathbb{Z}/2)$. By theorem B.6 it is enough to proof $Sq^0(\iota_n) = \iota_n$ for $\iota_n \in H^n(K(\mathbb{Z}/2, n))$ our canonical generator. Now we have the string of isomorphisms from UCT, dualising (homological finite!), Hurewicz and theorem B.4:

$$H^n(K(\mathbb{Z}/2, n)) \to H_n(K(\mathbb{Z}/2, n))^\vee \to H_n(K(\mathbb{Z}/2, n)) \leftarrow \pi_n(K(\mathbb{Z}/2, n)) \leftarrow H^n S^n$$

sending ι_n via $h(e^{K(\mathbb{Z}/2,n)})$ and $e^{K(\mathbb{Z}/2,n)}$ the unique generator of $\pi_n(K(\mathbb{Z}/2, n))$ to the generator $e^{\wedge n}$ of $H^n S^n$ and we only have to check that Sq^0 fixes $e^{\wedge n}$ but because Sq^0 commutes with suspensions, we only have to check it for $\tilde{e}^{\wedge 0} \in \tilde{H}^0(S^0)$ the generator of the reduced cohomology of S^0. But in this case $Sq^0(\tilde{e}^{\wedge 0}) = \left(\tilde{e}^{\wedge 0}\right)^2 = \tilde{e}^{\wedge 0}$. □

We finally have the result that if there is a natural commutative multiplication on C^*X, then $\mathrm{id}_{H^nX} = 0$ for all $n > 0$, i.e. every connected CW-complex has trivial cohomology. This is absurd. Because in all the proofs we have only used the fact that C^* is acyclic on models, the same holds for an arbitrary DGA over $\mathbb{F}_2$ quasi-isomorphic to C^*X. A more strict analysis of occurring signs (which we have ignored thanks to $\mathbb{F}_2$) would give these result for coefficients in every finite prime field.

References

[Bre93] Glen E. Bredon. *Topology and Geometry.* Vol. 139. Graduate texts in mathematics. New York: Springer, 1993.

[Cam15] Jonathan Campbell. “Rational Homotopy Theory”. Austin: University of Texas, 2015. 55 pp.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory.* eng. Vol. 205. Graduate texts in mathematics. New York, NY: Springer, 2001. 535 pp.

[GJ09] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* eng. Reprint of the 1999 ed. Modern Birkhäuser classics. Basel: Birkhäuser, 2009. 510 pp.

[Hes06] Kathrin Hess. “Rational Homotopy Theory: A Brief Introduction”. In: (2006).

[Hov07] Mark Hovey. *Model categories.* Vol. 63. Mathematical surveys and monographs. Providence, RI: American Mathematical Society, 2007.

[Leh14] Marina Christina Lehner. *All Concepts are Kan Extensions. Kan Extensions as the Most Universal of the Universal Constructions.* 2014. 27 pp.

[Lor15] Fosco Loregian. *This is the (co)end, my only (co)friend.* 2015.

[Mil89] Haynes Miller. “Notes on Steenrod Operations”. MIT, 1989. 105 pp.

[Moe15] Joshua Moermann. “Rational Homotopy Theory”. Nijmegen: Radboud University, 2015. 95 pp.

[Moe17] Ieke Moerdijk. “Rational Homotopy Theory”. Utrecht University, 2017. 53 pp.

[MT97] Ib H. Madsen and Jørgen Tornehave. *From calculus to cohomology. De Rham cohomology and characteristic classes.* eng. Reprint. Cambridge: Cambridge Univ. Press, 1997. 286 pp.

[Rie14] Emily Riehl. *Categorical Homotopy Theory.* New Mathematical Monographs. New York: Cambridge University Press, 2014.

Declaration of Academic Integrity

I hereby confirm that this thesis on *"Piecewise Linear Differential Forms"* is solely my own work and that I have used no sources or aids other than the ones stated. All passages in my thesis for which other sources, including electronic media, have been used, be it direct quotes or content references, have been acknowledged as such and the sources cited.

(date and signature of student)

I agree to have my thesis checked in order to rule out potential similarities with other works and to have my thesis stored in a database for this purpose.

(date and signature of student)